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Ma 221 Exam IB Solutions 14S

Solve the following equations:

1 [25 **pts.**]

$$\left(3x^2 + y^2 - 2\right)dx + (2xy + 6y - 3)dy = 0$$

Solution: If we let $M = 3x^2 + y^2 - 2$ and $N = 2xy + 6y - 3$, then

$$M_y = 2y = N_x$$

Hence this equation is exact. Thus there exist $f(x,y)$ such that

$$f_x = M = 3x^2 + y^2 - 2 \text{ and } f_y = N = 2xy + 6y - 3$$

Integrating f_x with respect to x leads to

$$f = x^3 + xy^2 - 2x + g(y)$$

Therefore

$$f_y = 2xy + g'(y) = N = 2xy + 6y - 3$$

We see that

$$g'(y) = 6y - 3$$

so

$$g(y) = 3y^2 - 3y + C$$

Hence, choosing $C = 0$

$$f = x^3 + xy^2 - 2x + 3y^2 - 3y$$

and the solution is given by

$$f = x^3 + xy^2 - 2x + 3y^2 - 3y = K$$

2 [25 **pts.**]

$$\frac{dy}{dx} = \frac{y\sin x}{1 + 2y^2}; \quad y(0) = 1$$

Solution: This equation is separable, since it may be rewritten as

$$\left(\frac{1 + 2y^2}{y}\right)dy = \sin x dx$$

or

$$\left(\frac{1}{y} + 2y\right)dy = \sin x dx$$

Integrating we have

$$\ln|y| + y^2 = -\cos x + C$$

The initial condition implies that y is not negative (so we drop the absolute value) and

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$$\ln 1 + (1)^2 = -1 + C$$

so $C = 2$ and the solution is

$$\ln y + y^2 + \cos x = 2$$

3 [25 **points**]

$$\cos^2 t \sin t y' = -\cos^3 t y + 1 \quad y\left(\frac{\pi}{4}\right) = 0$$

Solution:We may rewrite the equation as

$$y' + \frac{\cos t}{\sin t} y = \frac{1}{\cos^2 t \sin t}$$

This is a first order linear DE. The integrating factor is

$$e^{\int P(t)dt} = e^{\int \frac{\cos t}{\sin t} dt} = e^{\ln|\sin t|} = \sin t$$

Multiplying the DE by $\sin t$ we have

$$\sin t y' + \cos t y = \frac{1}{\cos^2 t}$$

or

$$\frac{d((\sin t)y)}{dt} = \frac{1}{\cos^2 t} = \sec^2 t$$

$$\sin t y = \tan t + c$$

The initial condition $y\left(\frac{\pi}{4}\right) = 0$ implies

$$0 = 1 + c$$

so $c = -1$ and the solution is given by

$$\sin t y = \tan t - 1$$

4 $\left[25\ \textbf{pts.}\right]$

$$\frac{dy}{dx} + \frac{1}{2} y = e^x y^3$$

Solution: This is a Bernoulli equation. We rewrite it as

$$y^{-3} \frac{dy}{dx} + \frac{1}{2} y^{-2} = e^x$$

Let $z = y^{-2}$. Then $z' = -2y^{-3}y'$ and we may write the above DE as

$$\frac{z'}{-2} + \frac{1}{2} z = e^x$$

or

$$z' - z = -2e^x$$

This is first order linear in z. The integrating factor is

$$e^{\int P dx} = e^{-\int dx} = e^{-x}$$

Multiplying the DE by e^{-x} we have

$$e^{-x} z' - e^{-x} z = -2$$

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or

$$\frac{d(e^{-x}z)}{dx} = -2$$

Integrating we have

$$e^{-x}z = -2x + c$$

Thus

$$z = y^{-2} = -2xe^{x} + ce^{x}$$

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Table of Integrals

$\int \sec^2 t dt = \tan t + C$
$\int \tan t dt = \ln(\sec t) + C$
$\int \frac{\sec^2 t}{\tan t} dt = \ln(\tan t) + C$
$\int t e^{at} dt = \frac{1}{a^2} e^{at}(at - 1) + C$
$\int t^2 e^{at} dt = \frac{1}{a^3} e^{at}\left(a^2 t^2 - 2at + 2\right) + C$
$\int \cos^2 t dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + C$
$\int \cos^3 t dt = \frac{1}{3} \cos^2 t \sin t + \frac{2}{3} \sin t + C$
$\int \sin^2 t dt = \frac{1}{2} t - \frac{1}{4} \pi - \frac{1}{4} \sin 2t + C$
$\int \sin^3 t dt = \frac{1}{12} \cos 3t - \frac{3}{4} \cos t + C$