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Ma 221 **Exam II A Solutions** **14S**

1. (30 **pts**. **total**) Consider the differential equation

$$L[y] = y'' - 3y' - 4y = 3e^{-x} + 4x^2.$$

1 a (6 **pts**.) Find the solution of the corresponding homogeneous equation $L[y] = 0$.

Solution: The characteristic equation is

$$p(r) = r^2 - 3r - 4 = (r-4)(r+1) = 0$$

Thus $r = -1, 4$ and

$$y_h = c_1 e^{-x} + c_2 e^{4x}$$

1 b (20 **pts**.) Find a particular solution of the equation

$$L[y] = y'' - 3y' - 4y = 3e^{-x} + 4x^2.$$

Solution: We first find y_{p_1} for $3e^{-x}$. Since $p'(r) = 2r - 3$, we see that $p(-1) = 0$ but $p'(-1) = -5 \neq 0$ so

$$y_{p_1} = \frac{3xe^{-x}}{-5}$$

$y'' - 3y' - 4y = 3e^{-x} + 4x^2$

To find y_{p_2} corresponding to $4x^2$ we let

$$y_{p_2} = Ax^2 + Bx + C$$

Then

$$y'_{p_2} = 2Ax + B$$
$$y''_{p_2} = 2A$$

Plugging into the DE we have

$$2A - 6Ax - 3B - 4Ax^2 - 4Bx - 4C = 4x^2$$

$$-4A = 4$$
$$-6A - 4B = 0$$
$$2A - 3B - 4C = 0$$

, Solution is: $\left[A = -1, B = \frac{3}{2}, C = -\frac{13}{8}\right]$

Thus

$$y_{p_2} = -x^2 + \frac{3}{2}x - \frac{13}{8}$$

Hence

$$y_p = y_{p_1} + y_{p_2} = \frac{3xe^{-x}}{-5} - x^2 + \frac{3}{2}x - \frac{13}{8}$$

1 c (4 **pts**.) Give a general solution of the equation

$$L[y] = y'' - 3y' - 4y = 3e^{-x} + 4x^2.$$

Solution:

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$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^{-x} + c_2 e^{4x} - \frac{3xe^{-x}}{5} - x^2 + \frac{3}{2}x - \frac{13}{8}$$

2 (25 **pts**) Find a *particular* solution of the differential equation

$$y'' + 4y = 4\cos 2t$$

Solution: Consider the companion equation

$$v'' + 4v = 4\sin 2t$$

Let $w = y + iv$ multiply the second equation by i and add it to the first equation to get

$$w'' + 4w = 4(\cos 2t + i\sin 2t) = 4e^{2it}$$

$p(r) = r^2 + 4 = 0$ yields $r = \pm 2i$. Thus $\cos 2t$ and $\sin 2t$ are homogeneous solutions. $p'(r) = 2r$ so $p'(2i) = 4i \neq 0$. Hence

$$w_p = \frac{4te^{2it}}{4i} = -it(\cos 2t + i\sin 2t)$$

$$y_p = \mathrm{Re}\, w_p = t\sin 2t$$

Note: One can get the same result by assuming

$$y_p(t) = At\cos 2t + Bt\sin 2t$$

$$y_p'(t) = A\cos 2t + B\sin 2t + 2Bt\cos 2t - 2At\sin 2t$$

$$y_p''(t) = 4B\cos 2t - 4A\sin 2t - 4At\cos 2t - 4Bt\sin 2t$$

Then the DE implies

$$y_p''(t) + 4y_p(t) = 4B\cos 2t - 4A\sin 2t = 4\cos 2t$$

Hence $B = 1$ and $A = 0$ and we get the same result as above.

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3 $\left(25 \text{ **pts**.}\right)$ Find a general solution of the equation

$$y'' - 2y' + y = e^x \ln x.$$

Solution: We first find the homogeneous solution. The characteristic equation is

$$r^2 - 2r + 1 = (r-1)^2 = 0$$

Thus $r = 1$ is a repeated root so

$$y_h = c_1 e^x + c_2 x e^x$$

To find a particular solution we use the Method of Variation of Parameters and let $y_1 = e^x$ and $y_2 = xe^x$

$$y_p = v_1 e^x + v_2 x e^x$$

Then the two equations for v_1' and v_2' are

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1 + v_2' y_2 = \frac{f}{a}$$

become

$$v_1' e^x + v_2' x e^x = 0$$
$$v_1' e^x + v_2'(e^x + xe^x) = e^x \ln x$$

or

$$v_1' + x v_2' = 0$$
$$v_1' + v_2'(1+x) = \ln x$$

Thus $v_1' = -x v_2'$ so the second equation implies

$$v_2' = \ln x$$

Then from the first equation we have

$$v_1' = -x \ln x$$

From the table of integrals below

$$v_2 = x(\ln x - 1)$$
$$v_1 = -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2$$

so

$$y_p = v_1 e^x + v_2 x e^x = \left(-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2\right)e^x + x^2 e^x(\ln x - 1)$$

Hence

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$$\begin{aligned} y &= y_h + y_p \\ &= c_1e^x + c_2xe^x + \left(-\frac{1}{2}x^2\ln x + \frac{1}{4}x^2\right)e^x + x^2e^x(\ln x - 1) \\ &= c_1e^x + c_2xe^x + \frac{1}{2}x^2e^x\ln x - \frac{3}{4}x^2e^x \end{aligned}$$

SNB check $y'' - 2y' + y = e^x\ln x$, Exact solution is: $\left\{C_5e^x - \frac{3}{4}x^2e^x + \frac{1}{2}x^2e^x\ln x + C_6xe^x\right\}$

4 (20 **pts**.) Solve

$$x^2y'' - 3xy' + 20y = 0$$

Solution: This is an Euler equation with $p = -3$ and $q = 20$. The characteristic equation for a solution of the form x^m is

$$m^2 + (p-1)m + q = m^2 - 4m + 20 = 0$$

Thus

$$m = \frac{4 \pm \sqrt{16 - 4(1)(20)}}{2} = \frac{4 \pm \sqrt{-64}}{2} = 2 \pm 4i$$

For complex roots $\alpha \pm \beta i$ the solution is

$$y = x^\alpha[C_1\cos(\beta\ln|x|) + C_2\sin(\beta\ln|x|)]$$

which becomes here since $\alpha = 2, \beta = 4$

$$y = x^2[C_1\cos(4\ln|x|) + C_2\sin(4\ln|x|)]$$

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Table of Integrals

$$\int \ln t\,dt = t(\ln t - 1) + C$$

$$\int t \ln t = \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2 + C$$

$$\int (\ln t)^2 dt = t\left(\ln^2 t - 2\ln t + 2\right) + C$$

$$\int \frac{\ln t}{t} dt = \frac{1}{2}\ln^2 t + C$$

$$\int \frac{(\ln t)^2}{t} dt + C = \frac{1}{3}\ln^3 t + C$$

$$\int \frac{1}{t \ln t} dt = \ln(\ln t) + C$$