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Ma 221 **Exam II B Solutions** **14S**

1. (30 **pts**. **total**) Consider the differential equation

$$L[y] = y'' + 3y' - 4y = 3e^x + 4x^2.$$

1 a (6 **pts**.) Find the solution of the corresponding homogeneous equation $L[y] = 0$.

Solution: The characteristic equation is

$$p(r) = r^2 + 3r - 4 = (r+4)(r-1) = 0.$$

Thus $r = 1, -4$ and

$$y_h = c_1 e^x + c_2 e^{-4x}.$$

1 b (20 **pts**.) Find a particular solution of this equation.

Solution: The d.e. is

$$y'' + 3y' - 4y = 3e^x + 4x^2.$$

We first find y_{p_1} for $L[y] = 3e^x$. Since $p'(r) = 2r + 3,$ we see that $p(1) = 0$ but $p'(1) = 5 \neq 0$ so

$$y_{p_1} = \frac{3xe^x}{5}.$$

To find y_{p_2} corresponding to $L[y] = 4x^2$ we let

$$y_{p_2} = Ax^2 + Bx + C.$$

Then

$$\begin{aligned} y_{p_2}' &= 2Ax + B \\ y_{p_2}'' &= 2A \end{aligned}$$

Plugging into the DE we have

$$2A + 6Ax + 3B - 4Ax^2 - 4Bx - 4C = 4x^2.$$

$$\begin{aligned} -4A &= 4 \\ 6A - 4B &= 0 \\ 2A + 3B - 4C &= 0 \end{aligned}$$

, The solution is: $\left[A = -1, B = -\frac{3}{2}, C = -\frac{13}{8}.\right]$

Thus

$$\begin{aligned} y_{p_2} &= -x^2 - \frac{3}{2}x - \frac{13}{8} \\ y_p &= y_{p_1} + y_{p_2} \\ &= \frac{3xe^x}{5} - x^2 - \frac{3}{2}x - \frac{13}{8} \end{aligned}$$

1 c (4 **pts**.) Give a general solution of the equation

$$L[y] = y'' - 3y' - 4y = 3e^{-x} + 4x^2.$$

Solution:

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^x + c_2 e^{-4x} + \frac{3xe^x}{5} - x^2 - \frac{3}{2}x - \frac{13}{8}$$

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2 (25 **pts**) Find a *particular* solution of the differential equation

$$y'' + 4y = 4\sin 2t.$$

Solution: Consider the companion equation

$$x'' + 4x = 4\sin 2t$$

Let $z = x + iy$. Multiply the first equation by i and add it to the second equation to get

$$z'' + 4z = 4(\cos 2t + i\sin 2t) = 4e^{2it}$$

$p(r) = r^2 + 4 = 0$ yields $r = \pm 2i$. Thus $\cos 2t$ and $\sin 2t$ are homogeneous solutions. $p'(r) = 2r$ so $p'(2i) = 4i \neq 0$. Hence

$$\begin{aligned} z_p &= \frac{4te^{2it}}{4i} = -it(\cos 2t + i\sin 2t) \\ &= t(\sin 2t - i\cos 2t) \end{aligned}$$

$$y_p = \operatorname{Im} z_p = -t\cos 2t$$

Note: One can get the same result by assuming

$$y_p(t) = At\cos 2t + Bt\sin 2t$$

$$y_p'(t) = A\cos 2t + B\sin 2t + 2Bt\cos 2t - 2At\sin 2t$$
$$y_p''(t) = 4B\cos 2t - 4A\sin 2t - 4At\cos 2t - 4Bt\sin 2t$$

Then the DE implies

$$y_p''(t) + 4y_p(t) = 4B\cos 2t - 4A\sin 2t = 4\sin 2t$$

Hence $A = -1$ and $B = 0$ and we get the same result as above.

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3 (25 **pts**.) Find a general solution of the equation

$$y'' + 2y' + y = e^{-x}\ln x$$

Solution: We first find the homogeneous solution. The characteristic equation is

$$r^2 + 2r + 1 = (r+1)^2 = 0$$

Thus $r = -1$ is a repeated root so

$$y_h = c_1e^{-x} + c_2xe^{-x}.$$

To find a particular solution we use the Method of Variation of Parameters and let $y_1 = e^{-x}$ and $y_2 = xe^{-x}$

$$y_p = v_1e^{-x} + v_2xe^{-x}$$

Then the two equations for v_1' and v_2' are

$$v_1'y_1 + v_2'y_2 = 0$$
$$v_1'y_1 + v_2'y_2 = \frac{f}{a}$$

become

$$v_1'e^{-x} + v_2'xe^{-x} = 0$$
$$-v_1'e^{-x} + v_2'(e^{-x} - xe^{-x}) = e^{-x}\ln x$$

or

$$v_1' + xv_2' = 0$$
$$-v_1' + (1-x)v_2' = \ln x$$

Thus $v_1' = -xv_2'$ so the second equation implies

$$v_2' = \ln x$$

Then from the first equation we have

$$v_1' = -x\ln x$$

From the table of integrals below

$$v_2 = x(\ln x - 1)$$
$$v_1 = -\frac{1}{2}x^2\ln x + \frac{1}{4}x^2$$

so

$$y_p = v_1e^{-x} + v_2xe^{-x} = \left(-\frac{1}{2}x^2\ln x + \frac{1}{4}x^2\right)e^{-x} + x^2e^{-x}(\ln x - 1)$$

Thus

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$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1 e^{-x} + c_2 x e^{-x} + \left(-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2\right)e^{-x} + x^2 e^{-x}(\ln x - 1) \\
&= c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2}x^2 e^{-x}\ln x - \frac{3}{4}x^2 e^{-x}
\end{aligned}
$$

SNB Check $y'' + 2y' + y = e^{-x}\ln x$, Exact solution is: $\left\{C_2 e^{-x} - \frac{1}{4e^x}\left(3x^2 - 2x^2\ln x\right) + C_3 x e^{-x}\right\}$

4 (20 **pts**.) Solve

$$x^2 y'' - 3xy' + 13y = 0$$

Solution: This is a Cauchy-Euler equation. The indicial (auxiliary) equation for a solution of the form x^r is

$$r(r-1) - 3r + 13 = r^2 - 4r + 13 = 0$$

Thus

$$r = \frac{4 \pm \sqrt{16 - 4(1)(13)}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

For those who dislike the quadratic formula, completing the square gives

$$
\begin{aligned}
r^2 - 4r + 13 &= 0 \\
r^2 - 4r + 4 + 9 &= 0 \\
r^2 - 4r + 4 &= -9 \\
(r-2)^2 &= -3^2 \\
r - 2 &= \pm 3i \\
r &= 2 \pm 3i.
\end{aligned}
$$

For complex roots $\alpha \pm \beta i$ the solution is

$$y = x^\alpha[C_1\cos(\beta\ln|x|) + C_2\sin(\beta\ln|x|)]$$

which becomes here since $\alpha = 2, \beta = 4$

$$y = x^2[C_1\cos(3\ln|x|) + C_2\sin(3\ln|x|)]$$

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Table of Integrals

$$\int \ln t dt = t(\ln t - 1) + C$$

$$\int t \ln t = \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2 + C$$

$$\int (\ln t)^2 dt = t\left(\ln^2 t - 2\ln t + 2\right) + C$$

$$\int \frac{\ln t}{t} dt = \frac{1}{2}\ln^2 t + C$$

$$\int \frac{(\ln t)^2}{t} dt + C = \frac{1}{3}\ln^3 t + C$$

$$\int \frac{1}{t \ln t} dt = \ln(\ln t) + C$$