

Name: _____

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Ma 221

Exam IIIA Solutions

14S

I pledge my honor that I have abided by the Stevens Honor System. _____

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a _____

#1b _____

#2a _____

#2b _____

#3 _____

#4 _____

Total Score _____

Note: A table of Laplace Transforms is given at the end of the exam.

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1a (10 pts.) Use the definition of the Laplace transform to determine the Laplace transform of $f(t) = e^{at}$, where a is a constant. $\mathcal{L}\{e^{at}\}$.

Solution: (This is Example 2 on page 354 of our text.)

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt \\ &= \lim_{N \rightarrow \infty} \int_0^N e^{-(s-a)t} dt = \lim_{N \rightarrow \infty} \left[-\frac{e^{-(s-a)t}}{s-a} \right]_0^N \\ &= \lim_{N \rightarrow \infty} \left[\frac{1}{s-a} - \frac{e^{-(s-a)N}}{s-a} \right] = \frac{1}{s-a} \quad \text{for } s > a \end{aligned}$$

1b (15 pts.) Determine

$$\mathcal{L}^{-1} \left\{ \frac{3s+2}{s^2+2s+10} \right\}$$

Solution: (This is Example 4 on page 369 of our text.)

$$\begin{aligned} \frac{3s+2}{s^2+2s+10} &= \frac{3s+2}{(s+1)^2+9} = \frac{3s+2}{(s+1)^2+3^2} \\ &= \frac{3(s+1)}{(s+1)^2+3^2} - \frac{1}{(s+1)^2+3^2} \\ &= \frac{3(s+1)}{(s+1)^2+9} - \frac{1}{3} \frac{3}{(s+1)^2+3^2} \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{3s+2}{s^2+2s+10} \right\} &= 3\mathcal{L}^{-1} \left\{ \frac{(s+1)}{(s+1)^2+3^2} \right\} - \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{3}{(s+1)^2+3^2} \right\} \\ &= 3e^{-t} \cos 3t - \frac{1}{3} e^{-t} \sin 3t \end{aligned}$$

2a (15 pts.) Consider the initial value problem

$$y'' + y = \sin 2t \quad y(0) = 2 \quad y'(0) = 1$$

Let $Y(s) = \mathcal{L}\{y\}(s)$. Use Laplace transforms to show that

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

Solution: Taking the Laplace transform of both sides of the DE we have

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin 2t\}$$

or letting $Y(s) = \mathcal{L}\{y\}(s)$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$$

Using the given initial conditions we have

$$(s^2+1)Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

Thus

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$$Y(s) = \frac{2}{(s^2 + 4)(s^2 + 1)} + \frac{2s}{s^2 + 1} + \frac{1}{s^2 + 1}$$

2b (15 pts.) Find the solution to the initial problem above, namely,

$$y'' + y = \sin 2t \quad y(0) = 2 \quad y'(0) = 1$$

by finding

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{(s^2 + 4)(s^2 + 1)} + \frac{2s}{s^2 + 1} + \frac{1}{s^2 + 1}\right\}$$

Solution: In order to invert this we need to do a partial fractions breakup of

$$\frac{2}{(s^2 + 4)(s^2 + 1)}.$$

There are two ways to do this.

Method I using complex variables.

$$\frac{2}{(s^2 + 4)(s^2 + 1)} = \frac{2}{(s - 2i)(s + 2i)(s - i)(s + i)} = \frac{A}{s - 2i} + \frac{B}{s + 2i} + \frac{C}{s - i} + \frac{D}{s + i}$$

Multiplying by $s - 2i$ and letting $s = 2i$ gives

$$A = \frac{2}{(4i)(i)(3i)} = -\frac{1}{6i}$$

Similarly

$$B = \frac{2}{(-4i)(-3i)(-i)} = \frac{1}{6i}$$

$$C = \frac{2}{-i(3i)(2i)} = \frac{1}{3i}$$

$$D = \frac{2}{-3i(i)(-2i)} = -\frac{1}{3i}$$

Thus

$$\frac{2}{(s^2 + 4)(s^2 + 1)} = -\frac{1}{6i} \frac{1}{s - 2i} + \frac{1}{6i} \frac{1}{s + 2i} + \frac{1}{3i} \frac{1}{s - i} - \frac{1}{3i} \frac{1}{s + i}$$

Therefore

$$y(t) = -\frac{1}{6i} \mathcal{L}^{-1}\left\{\frac{1}{s - 2i}\right\} + \frac{1}{6i} \mathcal{L}^{-1}\left\{\frac{1}{s + 2i}\right\} + \frac{1}{3i} \mathcal{L}^{-1}\left\{\frac{1}{s - i}\right\} - \frac{1}{3i} \mathcal{L}^{-1}\left\{\frac{1}{s + i}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}$$

$$\begin{aligned} y(t) &= -\frac{1}{6i} e^{2it} + \frac{1}{6i} e^{-2it} + \frac{1}{3i} e^{it} - \frac{1}{3i} e^{-it} + 2 \cos t + \sin t \\ &= -\frac{1}{6i} [\cos 2t + i \sin 2t] + \frac{1}{6i} [\cos 2t - i \sin 2t] + \frac{1}{3i} [\cos t + i \sin t] - \frac{1}{3i} [\cos t - i \sin t] + 2 \cos t + \sin t \\ &= -\frac{2}{6} \sin 2t + \frac{2}{3} \sin t + 2 \cos t + \sin t \\ &= -\frac{1}{3} \sin 2t + \frac{5}{3} \sin t + 2 \cos t \end{aligned}$$

Method II without complex variables: As above

$$Y(s) = \frac{2}{(s^2 + 4)(s^2 + 1)} + \frac{2s}{s^2 + 1} + \frac{1}{s^2 + 1}$$

We have to decompose $\frac{2}{(s^2+4)(s^2+1)}$.

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$s = 0$ yields

$$\frac{1}{2} = \frac{B}{4} + D$$

or

$$2 = B + 4D$$

$s = 1$ yields

$$\frac{2}{5(2)} = \frac{A+B}{5} + \frac{C+D}{2}$$

or

$$2 = 2A + 2B + 5C + 5D$$

$s = -1$ yields

$$\frac{2}{5(2)} = \frac{-A+B}{5} + \frac{-C+D}{2}$$

or

$$2 = -2A + 2B - 5C + 5D$$

$s = 2$ yields

$$\frac{2}{8(5)} = \frac{2A+B}{8} + \frac{2C+D}{5}$$

or

$$2 = 10A + 5B + 16C + 8D$$

Thus we have the following equations

$$2 = B + 4D$$

$$2 = 2A + 2B + 5C + 5D$$

$$2 = -2A + 2B - 5C + 5D$$

$$2 = 10A + 5B + 16C + 8D$$

, Solution is: $[A = 0, B = -\frac{2}{3}, C = 0, D = \frac{2}{3}]$

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1} = -\frac{2}{3} \frac{1}{s^2+4} + \frac{2}{3} \frac{1}{s^2+1}$$

$$y(t) = -\frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} \right\} + \frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} + 2 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+1} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\}$$

$$y(t) = -\frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{2}{s^2+4} \right\} + \frac{5}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} + 2 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+1} \right\}$$

$$= -\frac{1}{3} \sin 2t + \frac{5}{3} \sin t + 2 \cos t$$

SNB check:

$$y'' + y = \sin 2t$$

$$y(0) = 2$$

$$y'(0) = 1$$

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, Exact solution is: $\left\{2 \cos t + \frac{5}{3} \sin t - \frac{1}{3} \sin 2t\right\}$

3 (25 pts.) Find the first 5 nonzero terms of the power series solution about $x = 0$ for the DE:

$$y'' + 4xy' - 4y = 0$$

Be sure to give the recurrence relation.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

so

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

The differential equation \Rightarrow

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + 4 \sum_{n=1}^{\infty} a_n n x^n - 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

Combining the second and third summations we have

$$\sum_{n=1}^{\infty} a_n n(n-1) x^{n-2} + 4 \sum_{n=2}^{\infty} a_n (n-1) x^n - 4a_0 = 0$$

Shifting the first series by letting $n-2 = k$ or $n = k+2$ we have

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + 4 \sum_{n=1}^{\infty} a_n (n-1) x^n - 4a_0 = 0$$

Replacing n and k by m we have

$$\sum_{m=1}^{\infty} \{a_{m+2}(m+2)(m+1) + 4a_m(m-1)\} x^m + a_2(2)(1) - 4a_0 = 0$$

Thus

$$a_2 = 2a_0$$

and we have the recurrence relation

$$a_{m+2}(m+2)(m+1) + 4a_m(m-1) = 0 \quad \text{for } m = 1, 2, 3, \dots$$

or

$$a_{m+2} = -\frac{4(m-1)}{(m+2)(m+1)} a_m \quad \text{for } m = 1, 2, 3, \dots$$

Therefore

$$a_3 = 0$$

$$a_4 = -\frac{4(1)}{4(3)} a_2 = -\frac{2}{3} a_0$$

$$a_5 = 0$$

since $a_3 = 0$.

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$$a_6 = -\frac{4(3)}{6(5)}a_4 = \frac{4(3)(2)}{6(5)(3)}a_0 = \frac{4}{(5)(3)}a_0$$

Thus

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \left[1 + 2x^2 - \frac{2}{3}x^4 + \frac{4}{(5)(3)}x^6 + \dots \right] + a_1 x$$

SNB check: $y'' + 4xy' - 4y = 0$, Series solution is: $\{y(0) + xy'(0) + 2x^2y(0) - \frac{2}{3}x^4y(0) + \frac{4}{15}x^6y(0) + O(x^7)\}$

4 (25 pts.) Consider the boundary value problem

$$x^2y'' + 3xy' + \lambda y = 0 \quad y(1) = y(2) = 0.$$

Assuming the solution is of the form $y = x^m$ leads to

$$m = -1 \pm \sqrt{1 - \lambda}.$$

Consider only the case $1 - \lambda < 0$. Find the eigenvalues (λ) and the corresponding eigenfunctions for the boundary value problem for this case.

Solution:

Since $1 - \lambda < 0$, then $\lambda > 1$. Let $1 - \lambda = -\beta^2 \neq 0$. We have the complex roots $-1 \pm \beta i$ and

$$y(x) = x^{-1}[c_1 \cos(\ln \beta x) + c_2 \sin(\ln \beta x)]$$

$$y(1) = (1)^{-1}[c_1 \cos 0 + c_2 \sin 0] = c_1 = 0$$

Thus

$$y(x) = c_2 x^{-1} \sin(\beta \ln x)$$

$$y(2) = \frac{1}{2} c_2 \sin(\beta \ln 2) = 0$$

Thus

$$\beta \ln 2 = n\pi \quad n = 1, 2, 3, \dots$$

or

$$\beta = \frac{n\pi}{\ln 2} \quad n = 1, 2, 3, \dots$$

And finally

$$\lambda = 1 + \beta^2 = 1 + \left(\frac{n\pi}{\ln 2} \right)^2 \quad n = 1, 2, 3, \dots$$

are the eigenvalues with corresponding eigenfunctions

$$y(x) = a_n x^{-1} \sin\left(\frac{n\pi}{\ln 2} \ln x\right).$$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s) = \hat{f}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		