

Ma 221**Exam IIIB Solutions****14S**

- 1a (10 pts.)** Use the definition of the Laplace transform to determine the Laplace transform of $f(t) = e^{at}$, where a is a constant. $\mathcal{L}\{e^{at}\}$.

Solution: (This is Example 2 on page 354 of our text.)

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt \\ &= \lim_{N \rightarrow \infty} \int_0^N e^{-(s-a)t} dt = \lim_{N \rightarrow \infty} \left[-\frac{e^{-(s-a)t}}{s-a} \right]_0^N \\ &= \lim_{N \rightarrow \infty} \left[\frac{1}{s-a} - \frac{e^{-(s-a)N}}{s-a} \right] = \frac{1}{s-a} \quad \text{for } s > a \end{aligned}$$

- 1b (15 pts.)** Determine

$$\mathcal{L}^{-1} \left\{ \frac{3s+5}{s^2+4s+13} \right\}$$

Solution:

$$\begin{aligned} \frac{3s+5}{s^2+4s+13} &= \frac{3s+5}{(s+2)^2+9} = \frac{3s+5}{(s+2)^2+3^2} \\ &= \frac{3(s+2)}{(s+2)^2+3^2} - \frac{1}{(s+2)^2+3^2} \\ &= \frac{3(s+2)}{(s+2)^2+3^2} - \frac{1}{3} \frac{3}{(s+2)^2+3^2} \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{3s+5}{s^2+4s+13} \right\} &= 3\mathcal{L}^{-1} \left\{ \frac{(s+2)}{(s+2)^2+3^2} \right\} - \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{3}{(s+2)^2+3^2} \right\} \\ &= 3e^{-2t} \cos 3t - \frac{1}{3} e^{-2t} \sin 3t \end{aligned}$$

- 2a (15 pts.)** Consider the initial value problem

$$y'' + y = \cos 2t \quad y(0) = 2 \quad y'(0) = 1.$$

Let $Y(s) = \mathcal{L}\{y\}(s)$. Use Laplace transforms to show that

$$Y(s) = \frac{s}{(s^2+4)(s^2+1)} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}.$$

Solution: Taking the Laplace transform of both sides of the DE we have

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\cos 2t\}$$

or letting $Y(s) = \mathcal{L}\{y\}(s)$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{s}{s^2+4}$$

Using the given initial conditions we have

$$(s^2+1)Y(s) - 2s - 1 = \frac{s}{s^2+4}$$

Thus

$$Y(s) = \frac{s}{(s^2 + 4)(s^2 + 1)} + \frac{2s}{s^2 + 1} + \frac{1}{s^2 + 1}$$

2b (15 pts.) Find the solution to the initial problem above, namely,

$$y'' + y = \cos 2t \quad y(0) = 2 \quad y'(0) = 1$$

by finding

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 4)(s^2 + 1)} + \frac{2s}{s^2 + 1} + \frac{1}{s^2 + 1}\right\}$$

Solution: In order to invert this we need to do a partial fractions breakup of

$$\frac{s}{(s^2 + 4)(s^2 + 1)}.$$

There are two ways to do this.

Method I using complex variables.

$$\frac{s}{(s^2 + 4)(s^2 + 1)} = \frac{2}{(s - 2i)(s + 2i)(s - i)(s + i)} = \frac{A}{s - 2i} + \frac{B}{s + 2i} + \frac{C}{s - i} + \frac{D}{s + i}$$

Multiplication by $(s^2 + 4)(s^2 + 1)$ gives

$$s = A(s + 2i)(s - i)(s + i) + B(s - 2i)(s - i)(s + i) + C(s - 2i)(s + 2i)(s + i) + D(s - 2i)(s + 2i)(s - i)$$

Letting $s = 2i$ gives

$$A = \frac{2i}{(4i)(i)(3i)} = -\frac{1}{6}$$

Similarly

$$B = \frac{-2i}{(-4i)(-3i)(-i)} = -\frac{1}{6}$$

$$C = \frac{i}{-i(3i)(2i)} = \frac{1}{6}$$

$$D = \frac{-i}{-3i(i)(-2i)} = \frac{1}{6}$$

Thus

$$\frac{2}{(s^2 + 4)(s^2 + 1)} = -\frac{1}{6} \frac{1}{s - 2i} - \frac{1}{6} \frac{1}{s + 2i} + \frac{1}{6} \frac{1}{s - i} + \frac{1}{6} \frac{1}{s + i}$$

Therefore

$$y(t) = -\frac{1}{6} \mathcal{L}^{-1}\left\{\frac{1}{s - 2i}\right\} - \frac{1}{6} \mathcal{L}^{-1}\left\{\frac{1}{s + 2i}\right\} + \frac{1}{6} \mathcal{L}^{-1}\left\{\frac{1}{s - i}\right\} + \frac{1}{6} \mathcal{L}^{-1}\left\{\frac{1}{s + i}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}$$

$$\begin{aligned} y(t) &= -\frac{1}{6}e^{2it} - \frac{1}{6}e^{-2it} + \frac{1}{6}e^{it} + \frac{1}{6}e^{-it} + 2\cos t + \sin t \\ &= -\frac{1}{6}[\cos 2t + i \sin 2t] - \frac{1}{6}[\cos 2t - i \sin 2t] + \frac{1}{6}[\cos t + i \sin t] + \frac{1}{6}[\cos t - i \sin t] + 2\cos t + \sin t \\ &= -\frac{2}{6} \cos 2t + \frac{2}{6} \cos t + 2\cos t + \sin t \\ &= -\frac{1}{3} \cos 2t + \frac{7}{3} \cos t + \sin t \end{aligned}$$

Method II without complex variables: As above

$$Y(s) = \frac{s}{(s^2 + 4)(s^2 + 1)} + \frac{2s}{s^2 + 1} + \frac{1}{s^2 + 1}$$

We have to decompose $\frac{s}{(s^2+4)(s^2+1)}$.

$$\frac{s}{(s^2 + 4)(s^2 + 1)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 1}$$

Clearing the denominators yields

$$s = (As + B)(s^2 + 1) + (Cs + D)(s^2 + 4).$$

We equate the coefficient of each power of s, starting with the highest, s^3 .

$$s^3 : 0 = A + C$$

$$s^2 : 0 = B + D$$

$$s : 1 = A + 4C$$

$$1 : 0 = B + 4D$$

The result is two sets of two linear equations. The solution is $A = -\frac{1}{3}$, $B = 0$, $C = \frac{1}{3}$, $D = 0$.

$$\frac{s}{(s^2 + 4)(s^2 + 1)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 1} = -\frac{1}{3} \frac{s}{s^2 + 4} + \frac{1}{3} \frac{s}{s^2 + 1}$$

$$y(t) = -\frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4} \right\} + \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \right\} + 2 \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\}$$

$$\begin{aligned} y(t) &= -\frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4} \right\} + \frac{7}{3} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\} \\ &= -\frac{1}{3} \cos 2t + \frac{7}{3} \cos t + \sin t \end{aligned}$$

SNB check:

$$y'' + y = \cos 2t$$

$$y(0) = 2$$

$$y'(0) = 1$$

, Exact solution is: $\left\{ \frac{7}{3} \cos t + \sin t - \frac{1}{3} \cos 2t \right\}$

3 (25 pts.) Find the first 5 nonzero terms of the power series solution about $x = 0$ for the DE:

$$y'' + 3xy' - 3y = 0$$

Be sure to give the recurrence relation.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

so

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

The differential equation \Rightarrow

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + 3 \sum_{n=1}^{\infty} a_n n x^n - 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

Combining the second and third summations, after splitting out the $n = 0$ term, we have

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + 3 \sum_{n=1}^{\infty} a_n (n-1) x^n - 3a_0 = 0$$

Shifting the first series by letting $n-2 = k$ or $n = k+2$ and replacing n by k in the second series, we have

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) + 4 \sum_{n=1}^{\infty} a_k (k-1) x^k - 4a_0 = 0$$

Combining the series,(and again taking care with the $k = 0$ term) we have

$$\sum_{k=1}^{\infty} \{a_{k+2}(k+2)(k+1) + 3a_k(k-1)\} x^k + a_2(2)(1) - 4a_0 = 0$$

Thus

$$a_2 = 2a_0$$

and we have the recurrence relation

$$a_{k+2}(k+2)(k+1) + 3a_k(k-1) = 0 \quad \text{for } k = 1, 2, 3, \dots$$

or

$$a_{k+2} = -\frac{3(k-1)}{(k+2)(k+1)} a_m \quad \text{for } k = 1, 2, 3, \dots$$

Therefore

$$a_3 = 0$$

$$a_4 = -\frac{3(1)}{4(3)} a_2 = -\frac{1}{2} a_0$$

$$a_5 = 0$$

since $a_3 = 0$.

$$a_6 = -\frac{3(3)}{6(5)}a_4 = \frac{3(3)(1)}{6(5)(2)}a_0 = \frac{3}{20}a_0$$

Thus

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \left[1 + 2x^2 - \frac{1}{2}x^4 + \frac{3}{20}x^6 + \dots \right] + a_1 x$$

SNB check: $y'' + 4xy' - 4y = 0$, Series solution is: $\{y(0) + xy'(0) + 2x^2y(0) - \frac{2}{3}x^4y(0) + \frac{4}{15}x^6y(0) + O(x^7)\}$

4 (25 pts.) Consider the boundary value problem

$$x^2y'' + 3xy' + \lambda y = 0 \quad y(1) = y(e^2) = 0.$$

Assuming the solution is of the form $y = x^m$ leads to

$$m = -1 \pm \sqrt{1 - \lambda}.$$

Consider only the case $1 - \lambda < 0$. Find the eigenvalues (λ) and the corresponding eigenfunctions for the boundary value problem for this case.

Solution:

Since $1 - \lambda < 0$, then $\lambda > 1$. Let $1 - \lambda = -\beta^2 \neq 0$. We have the complex roots $-1 \pm \beta i$ and

$$y(x) = x^{-1}[c_1 \cos(\ln \beta x) + c_2 \sin(\ln \beta x)]$$

$$y(1) = (1)^{-1}[c_1 \cos 0 + c_2 \sin 0] = c_1 = 0$$

Thus

$$y(x) = c_2 x^{-1} \sin(\beta \ln x)$$

$$y(e^2) = \frac{1}{2}c_2 \sin(\beta \ln e^2) = 0$$

Thus

$$2\beta = n\pi \quad n = 1, 2, 3, \dots$$

or

$$\beta = \frac{n\pi}{2} \quad n = 1, 2, 3, \dots$$

And finally

$$\lambda = 1 + \beta^2 = 1 + \left(\frac{n\pi}{2}\right)^2 \quad n = 1, 2, 3, \dots$$

are the eigenvalues with corresponding eigenfunctions

$$y(x) = a_n x^{-1} \sin\left(\frac{n\pi}{2} \ln x\right).$$

Name: _____

Lecture Section ____

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s) = \hat{f}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		