Ma 221 16 Dec. 1996

Final Exam Solutions

1.

a)

Solve the equation $xy' + y = 3x^2$ (8)

$$\underbrace{x\frac{dy}{dx} + y}_{} = 3x^2$$

$$\frac{d}{dx}(xy) = 3x^2$$

integrate both sides

$$xy = \int 3x^2 dx$$

$$xy = x^3 + C$$

$$y = \frac{x^3 + C}{x}$$

b)

Solve $(3x^2 - y^2)dy - 2xy\,dx = 0$

(Hint: y^{-4} is an integrating factor.) (10)

$$\underbrace{(3x^2 - y^2)dy}_{N} - \underbrace{2xydx}_{M} = 0$$

$M_y = -2x$

Not exact

$N_x = 6x$

Multiplying the DE by y^{-4} yields

$$\left(3x^2y^{-4} - y^{-2}\right)dy - 2xy^{-3}dx = 0 \qquad \text{which is exact.}$$

$\exists$ F such that $F_x = M$ $F_y = N$

$$\Rightarrow \qquad F = \int -2xy^{-3}dx = -x^2y^{-3} + g(y)$$

$$F_y = 3x^2y^{-4} + g'(y) = 3x^2y^{-4} - y^{-2}$$

$$g'(y) = -y^{-2}$$

$$g(y) = y^{-1} + C$$

Solution is given implicitly by $-x^2y^{-3} + y^{-1} + C = 0$

c)

Solve the initial value problem $y' = e^{x+y}$ $y(1) = 0$ (7)

$$\frac{dy}{dx} = e^x e^y$$

$$dy = e^x e^y dx$$

$$e^{-y} dy = e^x dx$$

$\int e^{-y} dy = \int e^x dx$ or $-e^{-y} = e^x + C$

substituting

$-1 = e + C$ $C = -e - 1$

$$-e^{-y} = e^x - e - 1$$

2.

Find the general solution to the differential equations

a.

$y'' + y' - 6y = 3x + 4e^{2x}$ (13)

$p(r) = r^2 + r - 6 = 0$

$-3, 2$

$y_h(x) = C_1 e^{-3x} + C_2 e^{2x}$

$y'' + y' - 6y = 3x \qquad y_p = A_1 x + A_0$

$A_1 - 6A_1 x - 6A_0 = 3x$ so $A_1 = -\frac{1}{2}$ and $A_0 = -\frac{1}{12}$ so for the polynomial we have

$y_p = -\frac{1}{2}x - \frac{1}{12}A_0$

For $y'' + y' - 6y = 4e^{2x}$ we have since e^{2x} is a homogeneous solution ($p(2) = 0$, but $p'(2) = 5 \neq 0$)

$y_p = \frac{4xe^{2x}}{5}$. Thus

$$y = C_1 e^{-3x} + C_2 e^{2x} - \frac{1}{2}x - \frac{1}{12} + \frac{4xe^{2x}}{5}$$

b.

$x^2 y'' - 4xy' + 4y = x^4 + x^2$ (12)

This is an Euler equation with $p = -4 \qquad q = 4$

$m^2 + (p-1)m + q = 0$

$m^2 - 5m + 4 = (m-4)(m-1) = 0$, so x^4, x are the homogeneous solutions. $y_h = C_1 x + C_2 x^4$

We use variation of parameters to get a particular solution.

$y_p = v_1 x + v_2 x^4$

$v_1' y_1 + v_2' y_2 = v_1' x + v_2' x^4 = 0$

$v_1' y_1' + v_2' y_2' = v_1' + v_2'(4x^3) = \frac{x^4 + x^2}{x^2} = x^2 + 1$

$$W[y_1, y_2] = \begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix} = 3x^4 \qquad \Rightarrow v_1' = \frac{\begin{vmatrix} 0 & x^4 \\ x^2+1 & 4x^3 \end{vmatrix}}{3x^4} = \frac{-x^6 - x^4}{3x^4} = -\frac{1}{3}\left(x^2 + 1\right)$$

$$v_1 = \int -\frac{1}{3}\left(x^2+1\right)dx = -\frac{1}{3}\left(\frac{x^3}{3}+x\right)$$

$$v_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & x^2+1 \end{vmatrix}}{3x^4} = \frac{x^3+x}{3x^4} = \frac{1}{3}\left(\frac{1}{x}+\frac{1}{x^3}\right) \Rightarrow v_2 = \frac{1}{3}\left(\ln x - \frac{1}{2}\frac{1}{x^2}\right)$$

$$y_p = v_1 x + v_2 x^4 = -\frac{1}{3}\left(\frac{x^3}{3}+x\right)x + \frac{1}{3}\left(\ln x - \frac{1}{2}\frac{1}{x^2}\right)x^4 = -\frac{1}{9}x^4 - \frac{1}{3}x^2 + \frac{1}{3}x^4\ln x - \frac{1}{6}x^2$$

However, x^4 is a homogeneous solution, so we may ignore it in y_p. Thus $y_p = \frac{1}{3}x^4\ln x - \frac{1}{2}x^2$.

Finally $y = y_h + y_p = C_1 x + C_2 x^4 + \frac{1}{3}x^4 \ln x - \frac{1}{2}x^2$

3.

a)

Find $\quad L^{-1}\left\{\frac{s}{s^2-4s+9}\right\}$ (10)

$$\frac{s}{s^2-4s+9} = \frac{s}{(s-2)^2+5} = \frac{s-2}{(s-2)^2+5} + \frac{2}{(s-2)^2+5}$$

$$L^{-1}\left\{\frac{s}{s^2-4s+9}\right\} = e^{2t}\cos\sqrt{5}\,t + \frac{2}{\sqrt{5}}e^{2t}\sin\sqrt{5}\,t$$

b.

Use Laplace Transforms to solve the initial value problem

$$y'' - 3y' + 2y = e^{3t}, \qquad y(0) = 1, \qquad y'(0) = 0 \qquad (15)$$

$$s^2L\{y\} - sy(0) - y'(0) - 3sL\{y\} + 3y(0) + 2L\{y\} = \frac{1}{s-3}$$

$$\left(s^2-3s+2\right)L\{y\} = \frac{1}{s-3} + s - 3 = \frac{(s-3)^2+1}{s-3}$$

$$L\{y\} = \frac{s^2-6s+10}{(s-3)(s-2)(s-1)} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s-1}$$

$$s = 3 \Rightarrow \qquad \frac{9-18+10}{(1)(2)} = A \Rightarrow \qquad A = \frac{1}{2}$$

$$s = 2 \Rightarrow \qquad \frac{4-12+10}{(-1)(1)} = B \Rightarrow \qquad B = -2$$

$$s = 1 \Rightarrow \qquad \frac{1-6-10}{2(1)} = C \Rightarrow \qquad C = \frac{5}{2}$$

Therefore $L\{y\} = \dfrac{\frac{1}{2}}{s-3} + \dfrac{-2}{s-2} + \dfrac{\frac{5}{2}}{s-1}$

$$\Rightarrow \qquad y = \tfrac{1}{2}e^{3t} - 2e^{2t} + \tfrac{5}{2}e^{t}$$

4.

a.

Find $\quad L^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}$ (10)

$$\frac{1}{s\left(s^2+4\right)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$s = 0 \Rightarrow \qquad A = \tfrac{1}{4}$$

$$s = 1 \Rightarrow \qquad \tfrac{1}{5} = \tfrac{1}{4} + \frac{B+C}{5}$$

$$\left.\begin{aligned} s = 1 \Rightarrow &\qquad \tfrac{1}{5} = \tfrac{1}{4} + \frac{B+C}{5} \\ s = -1 \Rightarrow &\qquad -\tfrac{1}{5} = -\tfrac{1}{4} + \frac{-B+C}{5} \end{aligned}\right\} \Rightarrow \qquad c = 0 \Rightarrow \qquad \tfrac{1}{5} = \tfrac{1}{4} + \frac{B}{5}$$

$$\Rightarrow \qquad -\tfrac{1}{20} = \frac{B}{5} \Rightarrow \qquad B = -\tfrac{1}{4}$$

$$\Rightarrow \qquad L^{-1}\left\{\frac{\frac{1}{4}}{s} - \frac{1}{4}\frac{s}{s^2+4}\right\} = \tfrac{1}{4} - \tfrac{1}{4}\cos 2t$$

b.

The larger root of the indicial equation for the differential equation (15)
$4xy'' + 3y' + 3y = 0$ is $\frac{1}{4}$. Find the first four nonzero terms in the series solution near the regular singular point $x = 0$ corresponding to this root.

$$y = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}} \qquad y' = \sum_{n=0}^{\infty} a_n\left(n+\tfrac{1}{4}\right)x^{n-\frac{3}{4}}$$

$$y'' = \sum_{n=0}^{\infty} a_n\left(n+\tfrac{1}{4}\right)\left(n-\tfrac{3}{4}\right)x^{n-\frac{7}{4}}$$

$$\text{D.E.} \Rightarrow \qquad \sum_{n=0}^{\infty} 4a^n\left(n+\tfrac{1}{4}\right)\left(n-\tfrac{3}{4}\right)x^{n-\frac{3}{4}} + 3\sum_{n=0}^{\infty} a_n\left(n+\tfrac{1}{4}\right)x^{n-\frac{3}{4}} + 3\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}} = 0$$

$$\sum_{n=0}^{\infty} a_n\Big[\underbrace{(4n+1)\left(n-\tfrac{3}{4}\right) + 3n + \tfrac{3}{4}}_{4n^2 - 2n - \frac{3}{4} + 3n + \frac{3}{4}}\Big]x^{n-\frac{3}{4}} + 3\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}} = 0$$

$$k - \tfrac{3}{4} = n + \tfrac{1}{4} \Rightarrow \qquad n = k - 1$$

$$\sum_{n=1}^{\infty} a_n n(4n+1)x^{n-\frac{3}{4}} + 3\sum_{n=1}^{\infty} a_{k-1}x^{k-\frac{3}{4}} = 0$$

$$\Rightarrow \qquad a_m = -\frac{3a_{m-1}}{m(4m+1)} \qquad m = 1, 2, \ldots.$$

$$a_1 = -\frac{3 * a_0}{5} \qquad a_2 = \frac{-3 * a_1}{2(9)} = \frac{9 * a_0}{5(2)(9)}$$

$$a_3 = \frac{-3 * a_2}{3(13)} = -\frac{9 * a_0}{(10)(9)(13)}$$

$$y = x^{\frac{1}{4}} \sum a_n x^n = a_0 x^{\frac{1}{4}}[1 -] = a_0 x^{\frac{1}{4}}\left[1 - \frac{3}{5}x + \frac{9}{90}x^2 - \frac{9}{90(13)}x^3 + \ldots\right]$$

5.

a.

Find $\quad L\{t^3 u(t-2)\}$ (13)

$$e^{-2s}L\{(t+2)^3\}$$

$$e^{-2s}L\{t^3 + 6t^2 + 12t + 8\}$$

$$e^{-2s}\left(\frac{3!}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}\right)$$

b.

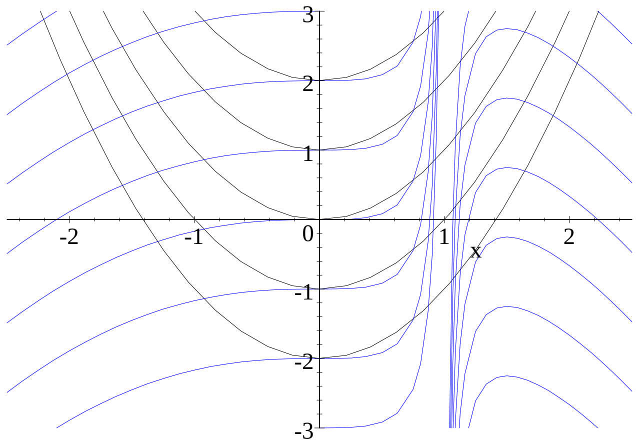
Use the method of isoclines to sketch the solution to

$$y' = y - x^2 \qquad y(0) = 1 \qquad (12)$$

Be sure to draw at least four isoclines.

let $\quad y - x^2 = c \quad \Rightarrow \quad y = x^2 + c$

direction field= $\quad y = \dfrac{x^3}{(3-3x)} + n$



6.

a.

Show that the equation $2xy'' + y' + xy = 0$ has a regular singularity at $x = 0$. Find the indicial equation, solve it, and state the form of the solution. Do not solve for the coefficients. (10)

$$y'' + \frac{1}{2x}y' + \frac{1}{2}y = 0 \qquad P = \frac{1}{2x} \qquad Q = \frac{1}{2}$$

P is not analytic at $x = 0 \quad \Rightarrow \quad x = 0 \quad$ is a singular point

$$xP = \frac{1}{2} \qquad x^2Q = \frac{x^2}{2} \qquad \text{both are analytic at } x = 0$$

$$\Rightarrow \qquad x = 0 \text{ is a regular singular point}$$

$$p_0 = \frac{1}{2} \qquad q_0 = 0 \qquad \Rightarrow \qquad m^2 + \left(\frac{1}{2} - 1\right)m = 0$$

$$\Rightarrow \qquad m^2 - \frac{1}{2}m = m\left(m - \frac{1}{2}\right) = 0 \qquad \Rightarrow \qquad m = 0, \quad \frac{1}{2}$$

Therefore $\quad y_1 = \sum_0^\infty a_n x^n \qquad y_2 = \sum_0^\infty b_n x^{n+\frac{1}{2}}$

b.

Find the power series solution of the equation $y'' + x^2y' + 2xy = 0$ near $x = 0$. Be sure to give the recurrence relation and the first 5 nonzero terms of the general solution. Indicate the two linearly independent solutions. (15)

$x = 0$ is an ordinary point $\quad \Rightarrow \quad y = \sum_0^\infty a_n x^n \qquad y' = \sum_1^\infty a_n (n) x^{n-1}$

$$y'' = \sum_2^\infty a_n (n)(n-1) x^{n-2}$$

$$\sum_{2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{1}^{\infty} a_n(n)x^{n+1} + 2\sum_{0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{1}^{\infty} a_n(n+2)x^{n+1} + 2a_0 x = 0$$

$$k+1 = n-2$$
$$n = k+3$$

$$\sum_{-1}^{\infty} a_{k+3}(k+3)(k+2)x^{k+1} + \sum_{1}^{\infty} a_n(n+2)x^{n+1} + 2a_0 x = 0$$

$$2a_0 x + a_2(2)(1)x^0 + a_3(3)(2)x + \sum_{1}^{\infty}\{a_{m+3}(m+3)(m+2) + a_m(m+2)\}x^{m+1} = 0$$

$$\Rightarrow \qquad a_2 = 0 \qquad 2a_0 + 6a_3 = 0 \qquad a_3 = -\frac{a_0}{3}$$

$$a_{m+3} = -\frac{a_m}{m+3} \qquad m = 1, 2, \ldots$$

$$a_4 = -\frac{a_1}{4} \qquad a_5 = 0 \qquad a_6 = -\frac{a_3}{6} = +\frac{a_0}{18} \qquad a_7 = -\frac{a_4}{7} = +\frac{a_1}{28}$$

$$y = a_0\left[1 - \tfrac{1}{3}x^3 + \tfrac{1}{18}x^6 + \ldots\right] + a_1\left[x - \tfrac{1}{4}x^4 + \tfrac{1}{28}x^7 + \ldots\right]$$

Part II: Choose any *two* questions.

7.

a.

Solve $(y^2 + 2xy)dx - x^2 dy = 0$ (10)

$$x^2 y' - 2xy = y^2$$

$$y' - \frac{2}{x}y = \frac{1}{x^2}\underbrace{y^2}_{\text{nonlinear term.}}$$

This is Bernoulli DE

DE$\Rightarrow$ $y^{-2}y' - \frac{2}{x}y^{-1} = \frac{1}{x^2}$

let $y^{-1} = v$

$$-y^{-2}y' = v'$$

DE$\Rightarrow$ $\quad v' - \frac{2}{x}v = \frac{1}{x^2}$ $\quad$ linear in v.

$$u = e^{-\int \frac{2}{x}dx} = x^{-2}$$

$$x^{-2}v' - 2x^{-3}v = x^{-4}$$

$$\frac{d}{dx}\left(x^{-2}v\right) = x^{-4}$$

$$x^{-2}v = -\tfrac{1}{3}x^{-3} + c$$

$$v = -\tfrac{1}{3}x^{-1} + cx^2$$

therefore

$$y = \frac{3x}{cx^3 - 1}$$

b.

Find the general solution of

$$y'' - 2y' - 3y = -10(\cos x + \sin x) \qquad (15)$$

$y'' - 2y' - 3y = -10\cos x$

$r'' - 2r' - 3r = -10\sin x$

$$p(x) = (x-3)(x+i)$$

$w'' - 2w' - 3w = -10e^{ix}$

$$p(x) = x^2 - 2x - 3 \qquad p(i) = -1 - 2i - 3 = -4 - 2i$$

$$w_p = \frac{-10e^{ix}}{-2(2+i)} = +\frac{5e^{ix}}{2+i} * \frac{2-i}{2-i} = \tfrac{5}{5}(2-i)[\cos x + i\sin x]$$

$y_p = [2\cos x + \sin x]$

$$\left.\begin{aligned} y_p &= [2\cos x + \sin x] \\ v_p &= [-\cos x + 2\sin x] \end{aligned}\right\}\text{total} \quad = \cos x + 3\sin x$$

therefore

$$y = C_1e^{3x} + C_2e^{-x} + \cos x + 3\sin x$$

8.

a.

Let $f(t) = \begin{cases} t & 0 \le t \le 1 \\ 0 & t \ge 1 \end{cases}$. Find $\mathrm{L}\{f(t)\}$ $\qquad$ (15)

$f(t) = t - u(t-1)t$

$$L\{f(t)\} = \frac{1}{s^2} - L\{u(t-1)t\}$$

$$= \frac{1}{s^2} - e^{-s}L\{t+1\} = \frac{1}{s^2} - \frac{e^{-5}}{s^2} - \frac{e^{-5}}{s}$$

$$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$$

b.

Show that the equation $(\frac{y^2}{2} + 2ye^z)dx + (y+e^x)dy = 0$ (10)

is *not* exact and then find an integrating factor for this equation. Do *not* solve the equation.

$$\frac{y^2}{2} + 2ye^x + (y+e^x)y' = 0$$

$$\Rightarrow \qquad \left(\frac{y^2}{2} + 2ye^x\right)dx + (y+e^x)dy = 0$$

$My = y + 2e^x \qquad Nx = e^x$ therefore not exact.

$$u\left(\frac{y^2}{2} + 2ye^x\right)dx + u(y+e^x)dy = 0$$

$$u_y\left(\frac{y^2}{2} + 2ye^x\right)dx + u(y+e^x)dy = u_x(y+e^x) + u(e^x)$$

$$u_y = 0 \qquad \Rightarrow \qquad u(y+e^x) = u_x(y+e^x)$$

$$u = u_x \qquad \Rightarrow \qquad u = e^x$$

$$\left(e^x\frac{y^2}{2} + 2ye^{2x}\right)dx + \left(ye^x + e^{2x}\right)dy = 0$$

$$My = ye^x + 2e^{2x}$$

$$Nx = ye^x + 2e^{2x}$$

9.

a.

Find the general solution of $y'' + 2y' + y = 6x^2e^{-x}$ (13)

$$\lambda^2 + 2\lambda + 1 = C\lambda + \lambda^2 = 0 \qquad \lambda = -1$$

$$y_h = C_1 e^{-x} + C_2 e^{-x}$$

$$y_p = Ax^2 e^{-x} + Bx^3 e^{-x} + Cx^4 e^{-x}$$

$$y_p' = 2Axe^{-x} - Ax^2 e^{-x} + 3Bx^2 e^{-x} - Bx^3 e^{-x} + 4Cx^3 e^{-x} - Cx^4 e^{-x}$$

$$y_p'' = 2Ae^{-x} - 4Axe^{-x} + Ax^2 e^{-x} + 6Bxe^{-x} - 6Bx^2 e^{-x} + Bx^3 e^{-x} + 12Cx^2 e^{-x} - 8Cx^3 e^{-x} + Cx^4 e^{-x}$$

$$\text{DE} \Rightarrow \qquad 2Ae^{-x} + 6Bxe^{-x} + 12Cx^2 e^{-x} = 6x^2 e^{-x}$$

$$A = 0 = B \qquad C = \tfrac{1}{2}$$

$$y_p - \tfrac{1}{2} x^4 e^{-x} \qquad y = C_1 e^{-x} + C_2 x e^{-x} + \tfrac{1}{2} x^4 e^{-x}$$

b.

Solve the equation $y(t) = 2t^2 + \int_0^t y(t-v) e^{-v} dv$ (12)

$$\text{L}\{y\} = \frac{4}{s^3} + \text{L}\{y\} \frac{1}{S+1}$$

$$\text{L}\{y\}\left[1 - \frac{1}{s+1}\right] = \frac{4}{s^3}$$

$$\text{L}\{y\}\left[\frac{s}{s+1}\right] = \frac{4}{s^3}$$

$$\text{L}\{y\} = \frac{4(s+1)}{s^4} = \frac{4}{s^3} + \frac{4}{s^4}$$

$$y = 2t^2 + \tfrac{2}{3} t^3$$