Ma 221

BOUNDARY VALUE PROBLEMS

Homogeneous Boundary Value Problems

Consider the following problem:

$$\left.\begin{array}{ll} \textbf{D.E.}\ L[y] = a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 & a \le x \le b \\ \text{B.C.}\ \alpha_1 y(a) + \beta_1 y'(a) = 0 & \alpha_1^2 + \beta_1^2 \ne 0 \\ \text{B.C.}\ \alpha_2 y(b) + \beta_2 y'(b) = 0 & \alpha_2^2 + \beta_2^2 \ne 0 \end{array}\right\} \qquad (1)$$

Here $\alpha_1, \alpha_2, \beta_1,$ and β_2 are constants.

Example

$$y'' = 0 \qquad y'(0) = y'(1) = 0$$

(Here $\alpha_1 = \alpha_2 = 0$)

$\Rightarrow y = Ax + b \qquad y'(x) = A \qquad y'(0) = y'(1) = A = 0$

$\Rightarrow y(x) = b \qquad b$ any constant.

The Boundary Value Problem (1) is called linear and homogeneous since if $u_1(x)$ and $u_2(x)$ satisfy it, $\Rightarrow c_1u_1(x) + c_2u_2(x)$ also does.

Example

$$y'' - 6y' + 5y = 0 \quad y(0) = 1 \quad y(2) = 1$$

Solution: The characteristic equation is

$$r^2 - 6r + 5 = (r-5)(r-1) = 0$$

so $r = 1, 5$

Thus

$$y(x) = c_1e^x + c_2e^{5x}$$

$$y(0) = c_1 + c_2 = 1$$
$$y(2) = c_1e^2 + c_2e^{10} = 1$$

Thus from the first equation $c_2 = 1 - c_1$ and the second equation becomes

$$c_1e^2 + (1-c_1)e^{10} = 1$$

$$c_1(e^2 - e^{10}) = 1 - e^{10}$$

$$c_1 = \frac{1-e^{10}}{e^2-e^{10}}$$

$$c_2 = 1 - \frac{1-e^{10}}{e^2-e^{10}} = \frac{1}{e^2-e^{10}}(e^2-1)$$

$$y = \frac{1-e^{10}}{e^2-e^{10}}e^x + \frac{e^2-1}{e^2-e^{10}}e^{5x}$$

SNB check

$$y'' - 6y' + 5y = 0$$
$$y(0) = 1$$
$$y(2) = 1$$

, Exact solution is: $\left\{\frac{e^{5x}}{e^2-e^{10}}(e^2-1) - \frac{e^x}{e^2-e^{10}}(e^{10}-1)\right\}$

Remark. The homogeneous Boundary Value Problem (B.V.P.) always possesses the solution $y(x) = 0$.

Question. When does there exist a nonzero solution to (1)?

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of $L[y] = 0$. $\Rightarrow y(x) = c_1y_1 + c_2y_2$ is the general solution of the DE.

B.C. $\Rightarrow$ $\left.\begin{array}{l} \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{array}\right\}$ and $y(x) = c_1y_1 + c_2y_2 \Rightarrow$

$$c_1[\alpha_1 y_1(a) + \beta_1 y_1'(a)] + c_2[\alpha_1 y_2(a) + \beta_1 y_2'(a)] = 0$$
$$c_1[\alpha_2 y_1(b) + \beta_2 y_1'(b)] + c_2[\alpha_2 y_2(b) + \beta_2 y_2'(b)] = 0.$$

The above are two equations for c_1 and c_2. We want a nontrivial solution. Let $B_a(u) = \alpha_1 u(a) + \beta_1 u'(a)$ and $B_b(u) = \alpha_2 u(b) + \beta_2 u'(b)$. Then the determinant of the coefficients of the above system must equal zero. Thus we require

$$\begin{vmatrix} B_a(y_1) & B_a(y_2) \\ B_b(y_1) & B_b(y_2) \end{vmatrix} = 0 \qquad (2)$$

Theorem 1. The homogeneous linear B.V.P. (1) has a nontrivial solution if and if (2) holds.

Theorem 2. If $u(x)$ is a particular nontrivial solution of the B.V.P. (1), then all solutions are given by $y = cu(x)$ where c is an arbitrary constant.

Proof. Let $v(x)$ be any solution, $u(x)$ a particular solution of the B.V.P. (1) $\Rightarrow \alpha_1 u(a) + \beta_1 u'(a) = 0$ and $\alpha_1 v(a) + \beta_1 v'(a) = 0$ since u and v both satisfy the first B.C. These equations may be regarded as equations for α_1, β_1. However, since by assumption α_1 and β_1 are not both zero $\Rightarrow$

$$\begin{vmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{vmatrix} = 0 = W[u,v]_{x=a} \Rightarrow W[u(x),v(x)] = 0 \text{ for } a \le x \le b$$

$\Rightarrow$ u and v are two LD solutions of the D.E. $\Rightarrow$ there exist constants $c_1, c_2 \neq 0$ such that $c_1u(x) + c_2v(x) = 0$ for $a \le x \le b \Rightarrow v(x) = -\frac{c_1}{c_2}u(x) = cu(x)$.

Example

$$y'' - \lambda^2 y = 0 \qquad \lambda \neq 0 \qquad y(0) = y(1) = 0$$

The general solution is $y = c_1e^{\lambda x} + c_2e^{-\lambda x}$. The B.C $y(0) = 0. \Rightarrow c_1 + c_2 = 0,$ whereas the condition $y(1) = 0$

leads to
$c_1 e^{\lambda} + c_2 e^{-\lambda} = 0.$ The two equations for c_1 and c_2 are

$$c_1 + c_2 = 0$$
$$c_1 e^{\lambda} + c_2 e^{-\lambda} = 0$$

The determinant of the coefficients is $\begin{vmatrix} 1 & 1 \\ e^{\lambda} & e^{-\lambda} \end{vmatrix} \neq 0. \Rightarrow c_1 = c_2 = 0 \Rightarrow$ the only solution is $y \equiv 0.$

Eigenvalue Problems

The following special kind of B.V.P. is called an eigenvalue problem.

$$\left.\begin{array}{ll} L[y] + \lambda y = 0 & a \le x \le b \\ \text{B.C. } \alpha_1 y(a) + \beta_1 y'(a) = 0 & \alpha_1^2 + \beta_1^2 \neq 0 \\ \text{B.C. } \alpha_2 y(b) + \beta_1 y'(b) = 0 & \alpha_2^2 + \beta_2^2 \neq 0 \end{array}\right\} (*)$$

Here $L[y] = a_0(x)y'' + a_1(x)y' + a_2(x)y,$ and λ is a parameter.

Again $y \equiv 0$ is a solution for all λ. However, we are interested in nontrivial (nonzero) solutions.

Definition. If a nontrivial solution of the B.V.P. $(*)$ exists for a value $\lambda = \lambda_i$, then λ_i is called an eigenvalue of L (relevant to the B.Cs.). The corresponding nontrivial solution $y_i(x)$ is called an eigenfunction.
Example Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \qquad y(1) = 0$$

We must consider three cases; $\lambda < 0, \lambda = 0,$, and $\lambda > 0$.

I. $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. Then the differential equation becomes

$$y'' - \alpha^2 y = 0$$

and has the general solution

$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$$

The boundary conditions $\Rightarrow$
$y'(0) = c_1\alpha - c_2\alpha = 0$ or $c_1 = c_2$, and $y(1) = c_1 e^{\alpha} + c_2 e^{-\alpha} = 0$.$\Rightarrow c_1 = c_2 = 0.$
Thus for $\lambda < 0$, the only solution is $y = 0$.

II. $\lambda = 0$. The solution is $y = c_1 x + c_2$. The BCs imply $c_1 = c_2 = 0$. Again the only solution is $y = 0$.
III. $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE becomes

$$y'' + \beta^2 y = 0$$

and has the general solution

$$y = c_1 \sin\beta x + c_2 \cos\beta x.$$

The BCs imply

$y'(0) = c_1\beta\cos 0 - c_2\beta\sin 0 = c_1\beta = 0.$ Hence $c_1 = 0,$, since $\beta \neq 0.$ Thus

$$y = c_2 \cos\beta x.$$

Now $y(1) = c_2 \cos\beta = 0$. Since we want a nontrivial solution we cannot have $c_2 = 0$. Hence

$$\cos\beta = 0 \Rightarrow \beta = \frac{2n+1}{2}\pi, n = 0, \pm 1, \pm 2, \ldots$$

We therefore have the eigenvalues

$$\lambda_n = \left(\frac{2n+1}{2}\right)^2 \pi^2,$$

and eigenfunctions

$$y_n(x) = C_n \cos\left(\frac{2n+1}{2}\right)x,$$

for $n = 0, 1, 2, \ldots$ Note the negative values of n do not give additional eigenfunctions since $\cos(-t) = \cos t$.

Example Find the eigenvalues and eigenfunctions for

$$y'' - 12y' + 4(7+\lambda)y = 0 \quad y(0) = y(5) = 0$$

Solution: The characteristic equation is

$$r^2 - 12r + 4(7+\lambda) = 0$$

so

$$r = \frac{+12 \pm \sqrt{144 - 4(4)(7+\lambda)}}{2} = 6 \pm 2\sqrt{2-\lambda}$$

Thus we have 3 cases to deal with, $2-\lambda < 0, 2-\lambda = 0,$ and $2-\lambda > 0.$

Case I: $2-\lambda > 0.$ Let $2-\lambda = \alpha^2$ where $\alpha \neq 0.$ The the general homogeneous solution is

$$y(x) = C_1 e^{(6+2\alpha)x} + C_2 e^{(6-2\alpha)x}$$

The BCs imply

$$C_1 + C_2 = 0$$
$$C_1 e^{(6+2\alpha)5} + C_2 e^{(6-2\alpha)5} = 0$$

, Solution is: $\{C_2 = 0, C_1 = 0\}.$ Thus $y = 0$ and there are no eigenvalues for this case.

Case II: $\lambda = 2.$ Then

$$y(x) = C_1 e^{6x} + C_2 x e^{6x}$$

The BCs imply

$$C_1 = 0$$
$$C_2(5)e^{30} = 0 \Rightarrow C_2 = 0$$

Therefore $\lambda = 2$ is not an eigenvalue.

Case III: $2-\lambda < 0.$ Let $2-\lambda = -\beta^2$ where $\beta \neq 0.$ Then $r = 6 \pm 2\beta i.$ The solution to the DE is

$$y(x) = C_1 e^{6x} \sin 2\beta x + C_2 e^{6x} \cos 2\beta x$$

The BCs imply

$$y(0) = C_2 = 0$$
$$y(5) = C_1 e^{30} \sin 10\beta = 0$$

Thus

$$10\beta = n\pi, \quad n = 1, 2, \ldots$$

or

$$\beta = \frac{n\pi}{10} \quad n = 1, 2, \ldots$$

and the eigenvalues are

$$\lambda = 2 + \beta^2 = 2 + \frac{n^2\pi^2}{100} \quad n = 1, 2, \ldots$$

The eigenfunctions are

$$y_n(x) = A_n e^{6x} \sin\left(\frac{n\pi}{5}\right)x$$

Example

$$y'' + \lambda y = 0 \qquad y(\pi) = y(2\pi) = 0$$

Solution: There are 3 cases to consider. $\lambda < 0, \lambda = 0,$, and $\lambda > 0$.

I. $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. Then the differential equation becomes

$$y'' - \alpha^2 y = 0$$

and has the general solution

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$$

Then

$$y(\pi) = c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = 0$$
$$y(2\pi) = c_1 e^{2\alpha\pi} + c_2 e^{-2\alpha\pi} = 0$$

Thus from the first equation

$$c_2 = -c_1 e^{2\alpha\pi}$$

and the second equation implies

$$c_1(e^{2\alpha\pi} - 1) = 0$$

Hence $c_1 = 0$ and thus $c_2 = 0$, so $y = 0$ is the only solution. There are no negative eigenvalues.

II. $\lambda = 0$. Then we have $y'' = 0$ so

$$y(x) = c_1 x + c_2$$

$$y(\pi) = c_1\pi + c_2 = 0$$
$$y(2\pi) = 2c_1\pi + c_2 = 0$$

Therefore $c_1 = c_2 = 0$ and $y = 0$, so 0 is not an eigenvalue.

III. $\lambda > 0$. Let $\lambda = \beta^2$ The DE becomes

$$y'' + \beta^2 y = 0$$

so

$$y(x) = c_1 \sin\beta x + c_2 \cos\beta x$$

The initial conditions yield

$$y(\pi) = c_1 \sin\beta\pi + c_2 \cos\beta\pi = 0$$
$$y(2\pi) = c_1 \sin 2\beta\pi + c_2 \cos 2\beta\pi = 0$$

This system will have a non-trivial solution if and only if

$$\begin{vmatrix} \sin\beta\pi & \cos\beta\pi \\ \sin 2\beta\pi & \cos 2\beta\pi \end{vmatrix} = 0$$

That is if and only if

$$\sin\beta\pi\cos 2\beta\pi - \cos\beta\pi\sin 2\beta\pi = \sin(\beta\pi - 2\beta\pi) = -\sin\beta\pi = 0$$

Thus we must have

$$\beta\pi = n\pi \quad n = 1,2,3,\ldots$$

or

$$\beta = n \quad n = 1,2,3,\ldots$$

Hence the eigenvalues are

$$\lambda = \beta^2 = n^2 \quad n = 1,2,3,\ldots$$

The two equations above for c_1 and c_2 become

$$c_1 \sin n\pi + c_2 \cos n\pi = 0$$
$$c_1 \sin 2n\pi + c_2 \cos 2n\pi = 0$$

Thus $c_2 = 0$ and c_1 is arbitrary. The eigenfunctions are

$$y_n(x) = a_n \sin nx$$

Remark. If $\vec{u}$ and $\vec{v}$ are 2 vectors, then $\vec{u} \perp \vec{v} \Leftrightarrow \vec{u} \cdot \vec{v} = 0$
$\vec{u} = (x_1,\ldots,x_n)$ $\quad \vec{v}(y_1,\ldots,y_n)$ As $n \to \infty$ $\quad \vec{u} \cdot \vec{v} \to \int x_i y_i$.

Definition. Let $f(x), g(x)$ be two continuous functions on $[a,b]$. We define the inner product of f and g in an interval $a \leq x \leq b$, denoted by $< f,g >$, by

$$< f,g >= \int_a^b f(x)g(x)dx.$$

Definition. Two functions f and g are said to be orthogonal on $[a,b]$ if

$$< f,g >= 0.$$

Example. $\int_0^\pi \sin x \cos x dx = \frac{\sin^2 x}{2}|_0^\pi = 0$ Therefore $\sin x$ and $\cos x$ are orthogonal on$[0,\pi]$.

Definition. The set of functions $\{f_1, f_2,\ldots\}$ is called an orthogonal set $< f_i, f_j >= 0 \quad i \neq j$.

Example. $\left\{1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots, \cos\frac{n\pi x}{L}, \ldots\right\}$ is an orthogonal set on $[0,L]$
Remark. For vectors we have the following: if $\vec{u} = (u_1,\ldots,u_n)$ then the length of $\vec{u} = \|\vec{u}\| = (\sum u_i^2)^{\frac{1}{2}} = \sqrt{\vec{u} \cdot \vec{u}}$. Motivated by this we have the following definition.

Definition. Let $f(x)$ be a continuous function on a$\leq x \leq b$. Then the norm of f is defined by

$$\|f\| = \sqrt{< f,f >} = \sqrt{\int_a^b f^2(x)dx}.$$

Example. $0 \leq x \leq 1 \quad \|x^2\|^2 =< x^2, x^2 >= \int_0^1 x^4 dx = \frac{x^5}{5}|_0^1 = \frac{1}{5}$
$\Rightarrow \|x^2\| = \frac{1}{\sqrt{5}}$.
Remark. Let $y = \frac{x^2}{\|x^2\|} = \frac{x^2}{\sqrt{5}} \Rightarrow \|y\| = \frac{\|x^2\|}{\sqrt{5}} = 1$.
Definition. If $\|f\| = 1$, then f is said to be normalized.

Definition. A set of functions $\{\phi_1, \phi_2,\ldots\}$ is called orthonormal if
(1) the set is orthogonal, and

(2) each has norm 1. Therefore $\{\phi_1, \phi_2, \dots\}$ is an orthonormal set $\Leftrightarrow$

$$< \phi_i, \phi_j >= \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Example $\{\sin(nx)\} = \{\sin x, \sin 2x, \sin 3x, \dots\}$ on$[0,\pi]$ is an orthogonal set since

$$< \sin(mx), \sin(nx) >= \int_0^\pi \sin mx \sin nx\, dx = \frac{1}{2}\int_0^\pi [\cos(m-n)x - \cos(m+n)x]dx \qquad m \neq n$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m-n} \right]_0^\pi$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n} \right] = 0 \quad m \neq n$$

since m and n are integers.
Now

$$< \sin nx, \sin nx >= \int_0^\pi \sin^2 nx dx$$

$$= \frac{1}{2}\int_0^\pi (1 - \cos 2nx)dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 2nx}{2n}\right)\Big|_0^\pi = \frac{\pi}{2}.$$

Therefore

$$\|\sin nx\| =< \sin nx, \sin nx >^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}$$

$\Rightarrow$ this set is not orthonormal. We can make an orthonormal set from these functions by dividing each element in the original by $\sqrt{\frac{\pi}{2}} \Rightarrow \left\{\sqrt{\frac{2}{\pi}} \sin nx\right\}$ is orthonormal set $(n = 1, 2, \dots)$.

Properties of the inner product.

$$1. < f, g >=< g, f > \text{ since } \qquad \int_a^b f(x)g(x)dx = \int_a^b g(x)f(x)dx$$

$$2. < \alpha f + \beta g, h >= \alpha < f, h > + \beta < g, h > \text{ since } \int(\alpha f + \beta g)dx = \alpha\int fdx + \beta\int gdx$$

$$3. a. < f, f >= 0 iff = 0$$
$$b. < f, f >> 0 iff \neq 0$$

Remarks. (1) It will be necessary when dealing with partial differential equations to "expand" an arbitrary function $f(x)$ in terms of an orthogonal set of functions $\{\psi_n\}$.

(2) Recall that in 3 space, if $\vec{u}_1 = (1,0,0), \vec{u}_2 = (0,1,0),$ and $\vec{u}_3 = (0,0,1)$ then $\vec{v} = (\alpha_1, \alpha_2, \alpha_3) = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$.
Note that

$$< \vec{u}_1, \vec{v} >= \vec{u}_1 \cdot \vec{v} =< \vec{u}_1, \alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 + \alpha_3\vec{u}_3 >=< \vec{u}_1, \alpha_1\vec{u}_1 > + < \vec{u}_1, \alpha_2\vec{u}_2 > + < \vec{u}_1, \alpha_3\vec{u}_3 >$$
$$= \alpha_1 < \vec{u}_1, \vec{u}_1 > + \alpha_2 < \vec{u}_1, \vec{u}_2 > + \alpha_3 < \vec{u}_1, \vec{u}_3 >= \alpha_1$$

Also $< \vec{u}_2, \vec{v} >= \alpha_2$ and $< \vec{u}_3, \vec{v} >= \alpha_3$.

Suppose we are given a set of orthogonal functions $\{\psi_n\}$ on $[0,L]$, and we desire to expand a function $f(x)$ given on $[0,L]$ in terms of them. Then we want

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \psi_n(x).$$

Question. What does α_k =?

Consider

$$\begin{aligned} &< \psi_k, f(x) > = < \psi_k, \sum_1^{\infty} \alpha_n \psi_n > \\ &= < \psi_k, \alpha_1\psi_1 + \alpha_2\psi_2 + \cdots > \\ &= \alpha_1 < \psi_k, \psi_1 > + \cdots + \alpha_k < \psi_k, \psi_k > + \alpha_{k+1} < \psi_k, \psi_{k+1} > + \cdots \end{aligned}$$

But $< \psi_k, \psi_j > = 0$ if $j \neq k$ since the set $\{\psi_k\}$ is orthogonal.

$\Rightarrow$

$$< \psi_k, f(x) > = \alpha_k < \psi_k, \psi_k > = \alpha_k \|\psi_k\|^2$$

Therefore

$$\alpha_k = \frac{\int_0^L f(x)\psi_k(x)dx}{\|\psi_k\|^2} = \frac{\int_0^L f(x)\psi_k(x)dx}{\int_0^L [\psi_k(x)]^2 dx} \qquad k = 1,2,\ldots \qquad (*)$$

$(*)$ is the formula for the coefficients in the expansion of a function $f(x)$ in terms of a set of orthogonal functions.

Ordinary Fourier Series

Fourier Sine Series

Consider the eigenvalue problem

$$D.E. y'' + \lambda y = 0 \qquad 0 \leq x \leq L \qquad B.C. y(0) = y(L) = 0$$

We shall first solve this problem. There are 3 cases to consider - $\lambda < 0, \lambda = 0, \lambda > 0$.

I. $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. The DE becomes

$$y'' - \alpha^2 y = 0$$

so

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

Then $y(0) = 0$ implies

$$c_1 + c_2 = 0$$

so $c_2 = -c_1$ and

$$y(x) = c_1[e^{\alpha x} - e^{-\alpha x}]$$

But then

$$y(L) = c_1[e^{\alpha L} - e^{-\alpha L}] = 0$$

So $c_1 = 0$ and hence $c_2 = 0$ and thus $y(x) = 0$ and there are no negative eigenvalues.

II. $\lambda = 0$ The the equation becomes $y'' = 0$ and $y = c_1 x + c_2$ and the BCs imply $y = 0$.

III. $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$ The DE becomes

$$y'' + \beta^2 y = 0$$

Thus

$$y = c_1 \sin \beta x + c_2 \cos \beta x$$

$y(0) = c_2 = 0$. Also

$$y(L) = c_1 \sin \beta L = 0$$

so

$$\beta = \frac{n\pi}{L} \quad n = 1, 2, 3, \ldots$$

$\Rightarrow$

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad n = 1, 2, 3, \ldots$$

are the eigenvalues, whereas the eigenfunctions are

$$\sin \sqrt{\lambda_n}\, x = \sin \frac{n\pi}{L} x = \psi_n \quad n = 1, 2, 3, \ldots$$

These functions form an orthogonal set.

Hence if

$$f(x) = \sum_1^\infty \alpha_k \sin \frac{k\pi x}{L}$$

then from $(*)$ above

$$\alpha_k = \frac{2}{L} \int_0^L f(x) \sin \frac{k\pi x}{L} dx,$$

since

$$\int_0^L [\psi_k(x)]^2 dx = \frac{L}{2}.$$

These formulas are for the Fourier *sine* series for $f(x)$ on $0 < x < L$.

Remarks. 1. At $x = 0$ and $x = L$ $\quad \sum \alpha_k \sin \frac{k\pi x}{L}$ gives 0 for $f(x)$. Therefore unless $f(0) = f(L) = 0$ the Fourier series is not good at the end points.

2. Since $\sin \frac{k\pi}{L}(x + 2L) = \sin\left(\frac{k\pi}{L}x + 2k\pi\right) = \sin \frac{k\pi x}{L}$, we see that the Fourier series yields $f(x + 2L) = f(x) \Rightarrow$ Fourier series has period $2L$. For $-L < x < 0$

we have $\sum_1^\infty \alpha_k \sin \frac{k\pi x}{L} = \sum_1^\infty \alpha_k \sin\left(\frac{-k\pi(-x)}{L}\right)$

$$= -\sum_1^\infty \alpha_k \sin \frac{k\pi(-x)}{L} \qquad -L < x < 0 \Rightarrow L > -x > 0$$

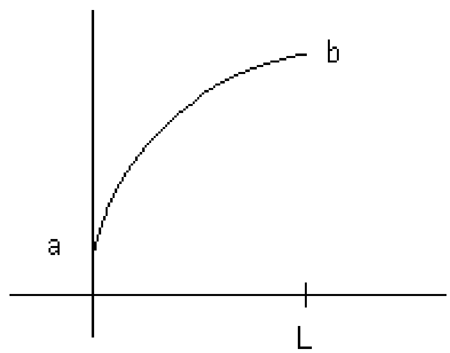
$= -f(-x),$ where $f(x)$ is value of series in $0 < x < L$.

Therefore the Fourier sine series converges to function $F(x)$ where

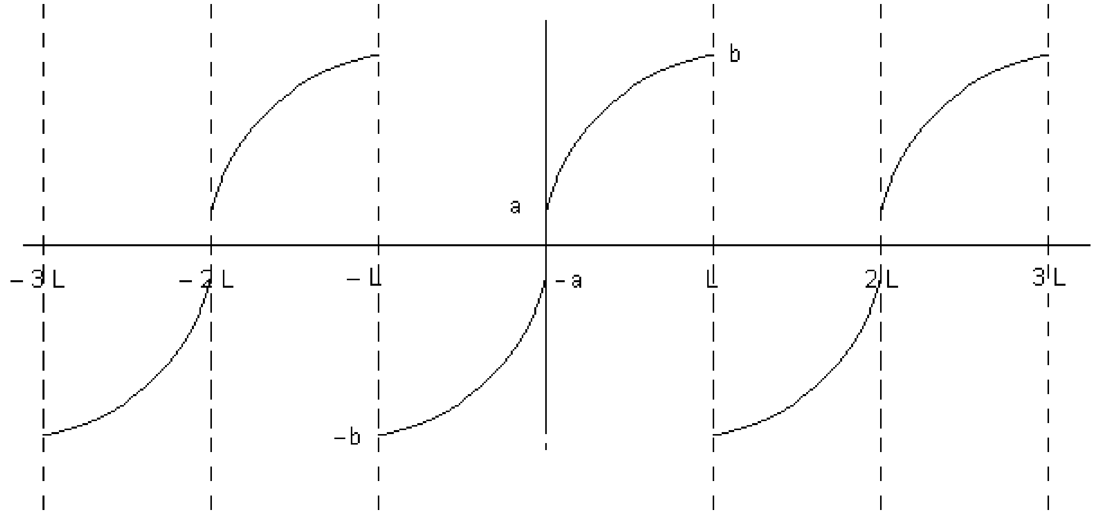
$$F(x) = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0 \end{cases} \qquad F(x+2L) = F(x)$$

This is the odd periodic extension of $f(x)$ with period $2L$. Unless $f(\pm kL) = 0$ $F(x)$ will be discontinuous at $\pm L, \pm 2L, \ldots$ Note that the function $f(x)$ is given on $[0, L]$ only, where the Fourier Sine series extends it to a function $F(x)$ which is define on $-\infty < x < \infty$.

Suppose that the graph of the function $f(x)$ is given by the figure below.

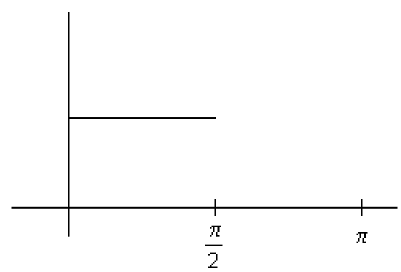


Then the Fourier sine series generates a function $F(x)$ defined on $-\infty < x < \infty$ whose graph is given below.



Example Find the Fourier sine series of

$$f(x) = \begin{cases} 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$



Now

$$f(x) = \sum \alpha_n \sin \frac{n\pi x}{L} = \sum_1^{\infty} \alpha_n \sin nx,$$

since $2L = 2\pi \Rightarrow L = \pi$.
The formula above for the coefficients in the Fourier sine series implies

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$\alpha_n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} 1 \cdot \sin nx dx + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} 0 \cdot \sin nx dx = -\frac{2}{\pi} \frac{\cos nx}{n} \Big|_0^{\frac{\pi}{2}}$$

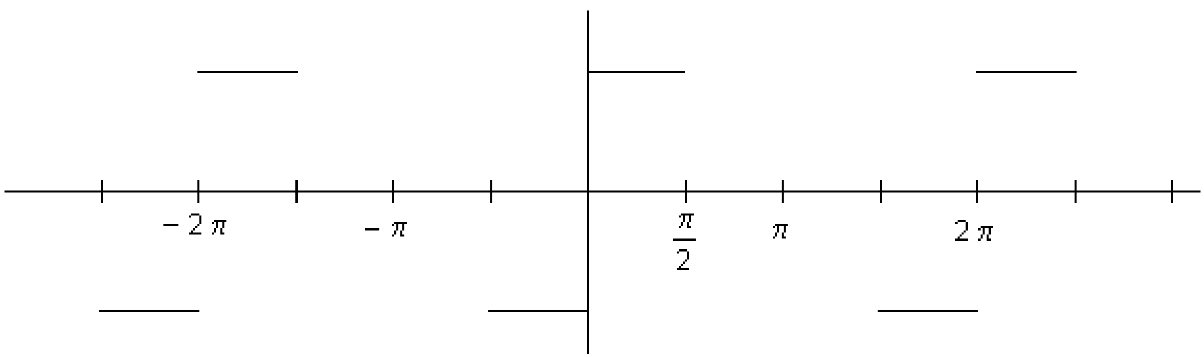
$$= -\frac{2}{\pi n} \left[\cos \frac{n\pi}{2} - 1 \right]$$

$$\alpha_n = \begin{cases} \dfrac{2}{\pi n} & n \text{ odd} \\ \left(\dfrac{-2}{\pi n}\right)\left[(-1)^{\frac{n}{2}} - 1\right] & n \text{ even} \end{cases}$$

Therefore

$$\begin{aligned} f(x) &= \sum_{1}^{\infty} \alpha_n \sin nx \\ &= \frac{2}{\pi}\left[\sin x + \frac{2}{2}\sin 2x + \frac{1}{3}\sin 3x + 0 \cdot \sin 4x + \frac{1}{5}\sin 5x + \frac{2}{6}\sin 6x + \cdots\right] \end{aligned}$$

Note that our function $f(x)$ on $0 \le x \le \pi$ is extended to the following on $-\infty < x < \infty$.



What we have done with *sine* functions can be done with *cosine* functions.

Fourier Cosine Series.

This comes from eigenvalue problem

$$\text{D.E.} \quad y'' + \lambda y = 0 \qquad \text{B.C.} \quad y'(0) = y'(L) = 0$$

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

are the eigenvalues and

$$\psi_n = \cos\frac{n\pi x}{L}$$

are the eigenfunctions, $n = 0, 1, 2, \ldots$.

Note $\lambda_0 = 0 \Rightarrow \psi_0 = 1$ which is an eigenfunction. Now we want to write

$$f(x) = \beta_0 + \sum_1^\infty \beta_n \cos\frac{n\pi x}{L}$$

Proceeding as above in our derivation of the constants in the Fourier Sine series, we get for the constants in the Fourier Cosine series

$$\beta_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad n = 1, 2, 3, \ldots \qquad \beta_0 = \frac{1}{L}\int_0^L f(x)dx$$

To see where the formula for β_0 comes from note

$$< \psi_0, f(x) >= \psi_0, \beta_0\psi_0 >=< 1, 1 > \beta_0$$

$$\Rightarrow \beta_0 = \frac{\int_0^L 1 \cdot f(x)dx}{\int_0^L 1^2 dx} = \frac{1}{L}\int_0^L f(x)dx.$$

Note the book writes

$$f(x) \sim \frac{a_0}{2} + \sum_1^\infty a_n \cos\frac{n\pi x}{L}$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad n - 0, 1, 2, \ldots$$

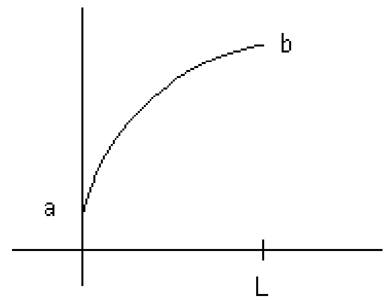
Thus

$$\beta_0 = \frac{a_0}{2}$$

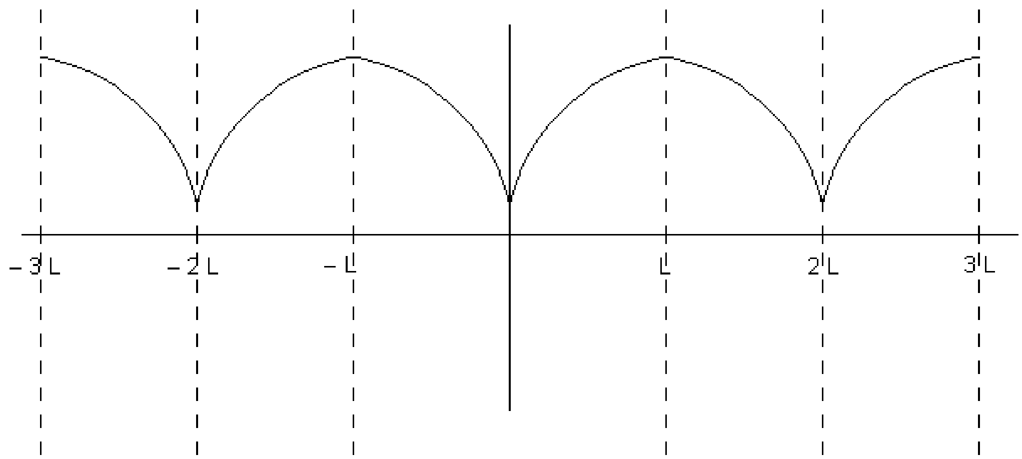
Again the Fourier series is periodic with period $2L$. However, now $f(-x) = f(x)$ since *cosine* is an even function. Here the Fourier Cosine series extends $f(x)$ which is given on $[0, L]$ to a function $F(x)$ which is defined on $-\infty < x < \infty$ as

$$F(x) = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0 \end{cases} \qquad F(x + 2L) = F(x).$$

If the graph of $f(x)$ looked as below



then $F(x)$, the *even* extension of $f(x)$, would look like



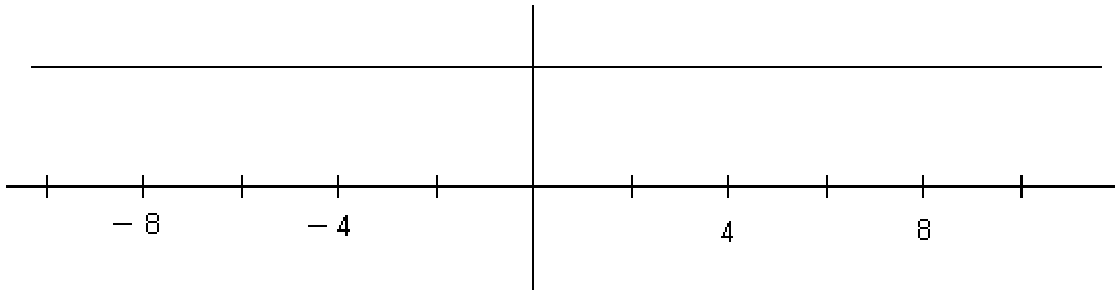
Example. Find the Fourier Cosine series for $f(x) = 1, \qquad 0 < x < 4$

$L = 4$

$$f(x) = \beta_0 + \sum_1^{\infty} \beta_n \cos \frac{n\pi x}{4} \qquad \beta_0 = \tfrac{1}{4}\int_0^4 f(x)dx = \tfrac{1}{4}\int_0^4 1 \cdot dx = 1$$

$$\beta_k = \tfrac{2}{4}\int_0^4 1 \cdot \cos \frac{n\pi x}{4} dx = \tfrac{1}{2}\left[\frac{\sin \frac{n\pi x}{4}}{\frac{n\pi}{4}}\right]_0^4 = \frac{2}{4\pi n}[\sin 0] = 0$$

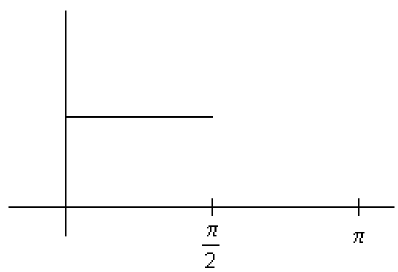
Therefore $f(x) = 1$ is its own Fourier Cosine series. The function is simply extended.



Example Find the Fourier cosine series of

$$f(x) = \begin{cases} 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

The graph of $f(x)$ is given below.



Note that this is the same function as in the previous example.
Now

$$f(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) = b_0 + \sum_{1}^{\infty} b_n \cos nx,$$

since the function is given on $[0, L] \Rightarrow L = \pi$.

$$b_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 1dx = \frac{1}{2}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1 \cdot \cos nxdx$$

$$= \frac{2}{n\pi}[\sin nx]_0^{\frac{\pi}{2}} = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

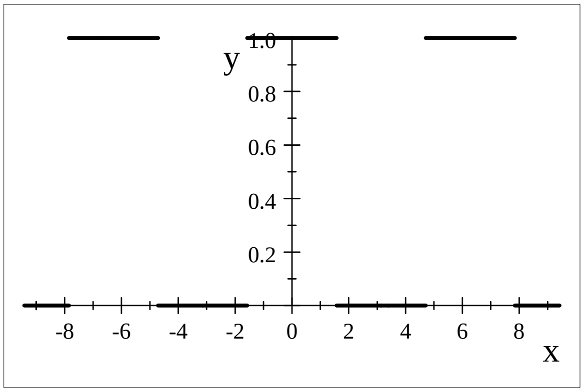
If n is even, then $\sin\left(\frac{n\pi}{2}\right) = 0.$ When n is odd, say $n = 2k+1, k = 0, 1, 2, \ldots$ then $\sin\left(\frac{n\pi}{2}\right) = \pm 1,$ depending on whether k is even or odd. Thus

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi}(-1)^k & n \text{ odd, } n = 2k+1, k = 0, 1, 2, \ldots \end{cases}$$

Thus

$$f(x) = b_0 + \sum_1^{\infty} b_n \cos nx = b_0 + b_1 \cos x + b_2 \cos 2x + \cdots$$
$$= \frac{1}{2} + \frac{2}{\pi}\cos x + 0\cos 2x - \frac{2}{2\pi}\cos 3x + 0\cos 4x + \frac{2}{5\pi}\cos 5x + \cdots$$

The graph of the even extension of the given function is



Example (a) Find the first four nonzero terms of the Fourier *cosine* series for the function

$$f(x) = x \text{ on } 0 < x < 1$$

Solution:

$$f(x) = \beta_0 + \sum_1^{\infty} \beta_n \cos\frac{n\pi x}{L}$$

where

$$\beta_0 = \frac{1}{L}\int_0^L f(x)dx \text{ and } \beta_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad n = 1,2,3,\ldots$$

Here $L = 1$ so

$$f(x) = \beta_0 + \sum_1^{\infty} \beta_n \cos n\pi x$$

$$\beta_0 = \frac{1}{1}\int_0^1 x dx = \frac{1}{2}$$
$$\beta_n = \frac{2}{1}\int_0^1 x\cos n\pi x dx = \frac{2}{n^2\pi^2}(\cos n\pi x + n\pi x \sin n\pi x)\Big|_0^1$$
$$= \frac{2}{n^2\pi^2}(\cos n\pi - 1) = \frac{2}{n^2\pi^2}((-1)^n - 1) \quad n = 1,2,3,\ldots$$

Hence $\beta_1 = -\frac{4}{\pi^2}$, $\beta_2 = 0$, $\beta_3 = -\frac{4}{9\pi^2}$, $\beta_4 = 0$, $\beta_5 = -\frac{4}{25\pi^2}$

Therefore

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\cos\pi x - \frac{4}{9\pi^2}\cos 3\pi x - \frac{4}{25\pi^2}\cos 5\pi x$$

Note: The book gives the formulas

$$f(x) = \frac{\beta_0}{2} + \sum_1^{\infty} \beta_n \cos\frac{n\pi x}{L}$$

where

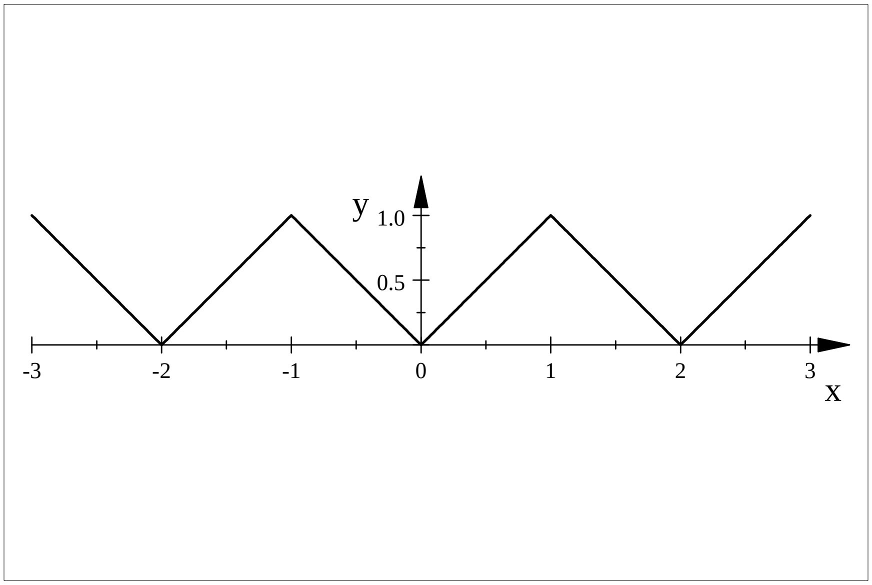
$$\beta_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad n = 0,1,2,3,\ldots$$

Using this formula we get

$$\beta_0 = \frac{2}{1}\int_0^1 x dx = 1$$

Therefore, the first term in the book's formula for the Fourier cosine series is $\frac{\beta_0}{2} = \frac{1}{2}$ as before.
(b) Sketch the graph of the function represented by the Fourier cosine series in (a) on $-3 < x < 3$.

x



Example (a) Find the Fourier *sine* series for the function

$$f(x) = x \text{ on } 0 < x < 1$$

Solution:

$$f(x) = \sum_1^\infty \alpha_k \sin\frac{k\pi x}{L}$$

where

$$\alpha_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}dx, \quad k = 1,2,3,\ldots$$

Here $L = 1$ so

$$f(x) = \sum_1^\infty \alpha_k \sin(k\pi x)$$

where

$$\alpha_k = 2\int_0^1 f(x)\sin(k\pi x)dx, \quad k = 1,2,3,\ldots$$

Thus

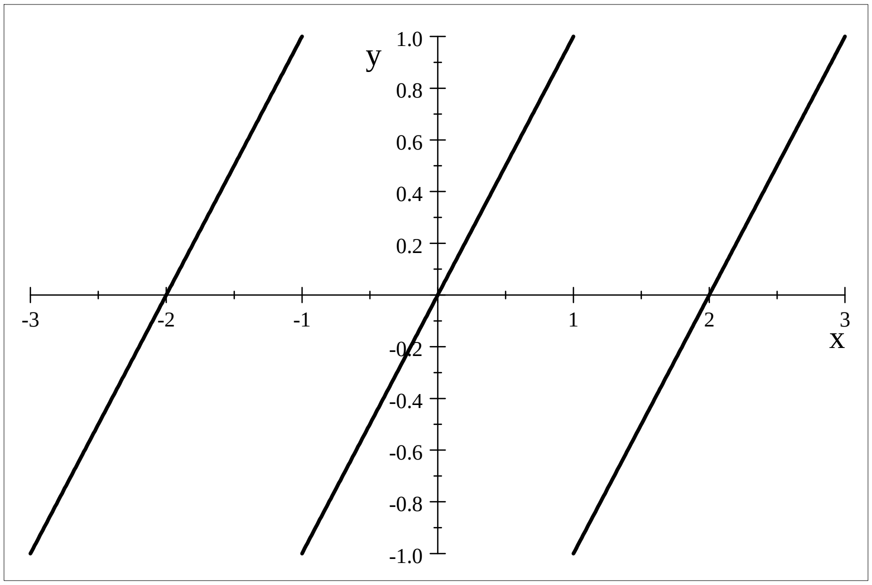
$$\alpha_k = 2\int_0^1 x\sin(k\pi x)dx = 2\left[\frac{1}{(k\pi)^2}(\sin k\pi x - k\pi x\cos k\pi x)\right]_0^1 =$$

$$= -2\left[\frac{1}{k\pi}\cos k\pi\right] = \frac{2}{k\pi}(-1)^{k+1} \quad k = 1,2,3,\ldots$$

Thus

$$f(x) = \frac{2}{\pi}\sum_{1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(k\pi x)$$

(b) Sketch the graph of the function represented by the Fourier *sine* series in 5 (a) on $-3 < x < 3$.
Solution:
1



Full Fourier Series (Omit)

This comes from the eigenvalue problem

$$D.E.\, y'' + \lambda y = 0 \qquad B.C.\, y(0) = y(2L) \qquad y'(0) = y'(2L) \qquad 0 \le x \le 2L$$

The eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

$n = 0, 1, 2, \ldots,$, whereas the eigenfunctions are

$$\psi_n = a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{2\pi x}{L} \qquad n = 0, 1, 2, ..$$

.
Note that for this problem the function $f(x)$ is given on $[0, 2L]$ since the eigenvalue problem is given on this interval. This is a different interval than that for Fourier Sine and Fourier Cosine series.

$\Rightarrow$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)dx, \qquad a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}dx \qquad b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx$$

Example Find full Fourier series for

$$f(x) = \begin{cases} 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$2L = \pi \Rightarrow L = \frac{\pi}{2}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 1 \cdot dx = \frac{1}{2}$$

$$a_n = \frac{1}{\frac{\pi}{2}}\int_0^{\pi} f(x)\cos\frac{n\pi x}{\frac{\pi}{2}}dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1 \cdot \cos 2nx\, dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos 2nx\, dx = \frac{2}{\pi}\frac{\sin 2nx}{2n}\Big|_0^{\frac{\pi}{2}} = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin 2nx\, dx = -\frac{2}{\pi}\frac{\cos 2nx}{2n}\Big|_0^{\frac{\pi}{2}} = \frac{1}{\pi n}[\cos n\pi - \cos 0] \qquad n = 1, 2, \ldots$$

$$b_n = -\frac{1}{\pi n}[(-1)^n - 1] = \begin{cases} +\frac{2}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

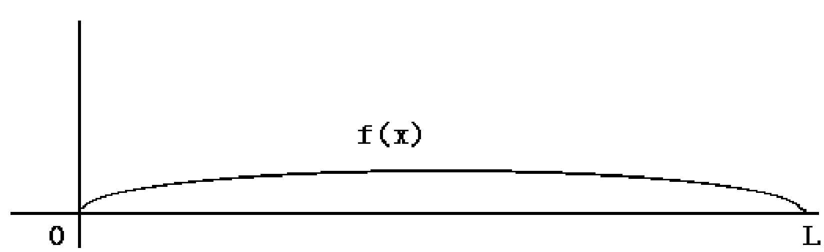
$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin 2x + \frac{1}{3}\sin 6x + \frac{1}{5}\sin 10x + \ldots\right]$$

The Vibrating String

It may be shown that the equation governing a string of length L vibrating is

$$y_{xx}(x,t) = \frac{\partial^2 y}{\partial x^2} = \frac{1}{\alpha^2} y_{tt}(x,t) \qquad (1)$$

Equation (1) is called the wave equation. Suppose string is held fixed at the ends $x = 0$ and $x = L$



$\Rightarrow$

$$(2a) \quad y(0,t) = 0 \qquad t \geq 0 \qquad B.C.$$

$$(2b) \quad y(L,t) = 0 \qquad t \geq 0 \qquad B.C.$$

Also suppose at $t = 0$ the string has displacement $y = f(x)$ and is released from rest

$\Rightarrow$

$$(3a) \quad y(x,0) = f(x) \qquad 0 \leq x \leq L \qquad I.C.$$

$$(3b) \quad y_t(x,0) = 0 \qquad 0 \leq x \leq L \qquad I.C.$$

In order to solve the above problem we shall assume $y(x,t) = X(x)T(t)$ separation of variables
$\Rightarrow y_x = X'T \qquad y_{xx} = X''T \qquad y_{tt} = XT''$. Note that $X', T', \ldots$ are ordinary derivatives of X with respect to x and T with respect to $t.$ Now the P.D.E. (1)

$\Rightarrow$

$$X''T = \frac{1}{\alpha^2} XT''$$

$\Rightarrow$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T''}{T}$$

.

Note that the left hand side is a function of x only, whereas the right hand side is a function of t only. This implies that each side must equal the same constant. Therefore

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T''}{T} = k$$

Hence we get the two ordinary differential equations

$$X'' - kX = 0 \qquad and \qquad T'' - \alpha^2 kT = 0$$

Now $y(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0,$ whereas $y(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0$. Therefore we must solve the problem

$$X'' - kX = 0 \qquad X(0) = X(L) = 0.$$

There are three cases. If $k = 0 \Rightarrow X \equiv 0$. If $k > 0 \Rightarrow X = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$.
and the boundary conditions $\Rightarrow c_1 = c_2 = 0.$

For the case $k < 0,$ let $k = -\lambda^2$
$\Rightarrow$

$$X'' + \lambda^2 X = 0 \qquad X(0) = X(L) = 0$$

This is an eigenvalue problem. The solution to the DE is

$$X = c_1 \sin \lambda x + c_2 \cos \lambda x$$

$X(0) = 0 \Rightarrow c_2 = 0$ whereas $X(L) = 0 \ \Rightarrow c_1 \sin \lambda = 0 \ \Rightarrow \lambda = \frac{n\pi}{L}$ for $n = \pm 1, \pm 2, \pm 3, \ldots$.
Since $\sin(-x) = -\sin x$ we may disregard the negative values of n.
Therefore

$$X_n(x) = c_n \sin \frac{n\pi}{L} x \qquad n = 1, 2, 3, \ldots$$

For $T(t)$ we have the equation

$$T'' + \alpha^2 \lambda^2 T = 0,$$

since $k = -\lambda^2$. Thus

$$T_n(t) = c \sin \alpha\lambda t + d \cos \alpha\lambda t = a_n \sin \frac{n\pi\alpha}{L} t + b_n \cos \frac{n\pi\alpha t}{L}.$$

But $y_t(x,0) = 0 \Rightarrow T'(0) = 0.$ Now $T'(t) = a_n\left(\alpha \frac{n\pi}{L}\right) \cos \alpha \frac{n\pi}{L} t - b_n \left(\alpha \frac{n\pi}{L}\right) \sin \alpha \frac{n\pi t}{L}$, so $T'(0) = 0$ $\Rightarrow a_n = 0$ for all n .
Therefore

$$T_n(t) = b_n \cos \frac{n\pi\alpha t}{L},$$

and we have finally that

$$y_n(x,t) = X_n(x) T_n(t) = c_n \sin \frac{n\pi x}{L} \times b_n \cos \frac{n\pi\alpha t}{L}$$

Let $c_n \times b_n = d_n$.
We note that

$$y_n(x,t) = d_n \sin \frac{n\pi x}{L} \cos \frac{n\pi\alpha t}{L} \quad n = 1, 2, 3, \ldots$$

satisfies the P.D.E. $y_{xx} = \frac{1}{\alpha^2} y_{tt}$ (1) and the boundary conditions $y(0,t) = y(L,t) = 0$ $(2a, 2b)$, as well as the initial condition $y_t(0) = 0$ $(3b)$.

What about the condition $y(x,0) = f(x)$? Notice that

$$y(x,t) = \sum_{1}^{\infty} d_n \sin \frac{n\pi x}{L} \cos \frac{n\pi\alpha t}{L}$$

is also a solution since of $(1), (2a, b)$ and $(3b)$. Thus $y(x,t)$ is solution of everything except condition $(3a)$, namely, $y(x,0) = f(x)$.
But

$$y(x,0) = \sum_{1}^{\infty} d_n \sin \frac{n\pi x}{L} = f(x).$$

Therefore if f has a Fourier sine series expansion we let

$\Rightarrow$

$$d_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Now with these coefficients d_n

$$y(x,t) = \sum_{1}^{\infty} d_n \sin \frac{n\pi x}{L} \cos \frac{n\pi\alpha t}{L}$$

is a solution to entire problem $(1), (2a, 2b), (3a, 3b)$.

Example

$$y_{xx} = y_{tt} \qquad y(0,t) = y(L,t) = 0$$
$$y_t(x,0) = 0$$
$$y(x,0) = 2\sin\frac{\pi x}{L}$$

Here $\alpha = 1$ and $f(x) = 2\sin\frac{\pi x}{L}$

Now

$$y(x,t) = \sum_{1}^{\infty} d_n \sin \frac{n\pi x}{L} \cos \frac{n\pi t}{L}$$

$$d_n = \frac{2}{L}\int_0^L \sin\frac{\pi x}{L} \sin\frac{n\pi x}{L} dx = \frac{2}{L}\int_0^L 2\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L} dx = 0 \qquad n = 2, 3, \ldots$$

$$d_1 = \frac{2}{L}\int_0^L (2)\sin^2\frac{n\pi x}{L} dx = \frac{4}{L}\left[\int_0^L\left(\frac{1-\cos\frac{2n\pi x}{L}}{2}\right)\right]dx = \frac{4}{L}\left[\frac{x}{2} - \left(\frac{\sin 2\frac{n\pi x}{L}}{\frac{2n\pi}{L}}\right)\right]_0^L = 2$$

$\Rightarrow$ solution is

$$y(x,t) = 2\sin\frac{n\pi}{L}\cos\frac{\pi t}{L}.$$

Example Solve:

$$\text{P.D.E.: } u_{xx} - 16u_{tt} = 0$$
$$\text{B.C.'s: } u(0,t) = 0 \qquad u_x(1,t) = 0$$
$$\text{I.C.: } u(x,0) = -3\sin\frac{5\pi x}{2} + 23\sin\frac{11\pi x}{2}; \quad u_t(x,0) = 0$$

Solution: We assume

$$u(x,t) = X(x)T(t)$$

The PDE implies

$$\frac{X''}{X} = 16\frac{T''}{T} = k \;\; k \text{ a constant}$$

Then we have the two ordinary DEs

$$X'' - kX = 0$$
$$T'' - \frac{1}{16}kT = 0$$

The boundary conditions for $X(x)$ are

$$X(0) = X'(1) = 0$$

so that the eigenvalue problem for X is

$$X'' - kX = 0 \quad X(0) = X'(1) = 0$$

For nontrivial solutions we let $k = -\beta^2, \beta \neq 0$ and get

$$X'' + \beta^2 X = 0$$

so

$$X(x) = C_1\sin\beta x + C_2\cos\beta x$$

$$X(0) = 0 \Rightarrow C_2 = 0$$

Thus

$$X'(x) = C_1\beta\cos\beta x$$

and $X'(1) = 0 \Rightarrow$

$$\beta = \left(\frac{2n+1}{2}\right)\pi \quad n = 0,1,2,\ldots$$

Therefore

$$X_n(x) = a_n\sin\left(\frac{2n+1}{2}\right)\pi x \quad n = 0,1,2,\ldots$$

Since

$$k = -\beta^2 = \left(\frac{2n+1}{2}\right)^2\pi^2$$

The equation for $T(t)$ becomes

$$T'' + \frac{1}{16}\left(\frac{2n+1}{2}\right)^2\pi^2 T = 0$$

so

$$T_n(t) = b_n\sin\left(\frac{2n+1}{8}\right)\pi t + c_n\cos\left(\frac{2n+1}{8}\right)\pi t \quad n = 0,1,2,\ldots$$

The BC $u_t(x,0) = 0 \Rightarrow T'(0) = 0.$ Since

$$T_n'(t) = b_n\left(\frac{2n+1}{8}\right)\pi\cos\left(\frac{2n+1}{8}\right)\pi t - c_n\left(\frac{2n+1}{8}\right)\pi\sin\left(\frac{2n+1}{8}\right)\pi t$$

we see that $b_n = 0$ so that

$$T_n(t) = c_n\cos\left(\frac{2n+1}{8}\right)\pi t \quad n = 0,1,2,\ldots$$

Thus

$$u_n(x,t) = X_n(x)T_n(t) = D_n \sin\left(\frac{2n+1}{2}\right)\pi x \cos\left(\frac{2n+1}{8}\right)\pi t \quad n = 0,1,2,\ldots$$

Let

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} D_n \sin\left(\frac{2n+1}{2}\right)\pi x \cos\left(\frac{2n+1}{8}\right)\pi t$$

Then

$$u(x,0) = \sum_{n=0}^{\infty} D_n \sin\left(\frac{2n+1}{2}\right)\pi x = -3\sin\frac{5\pi x}{2} + 23\sin\frac{11\pi x}{2}$$

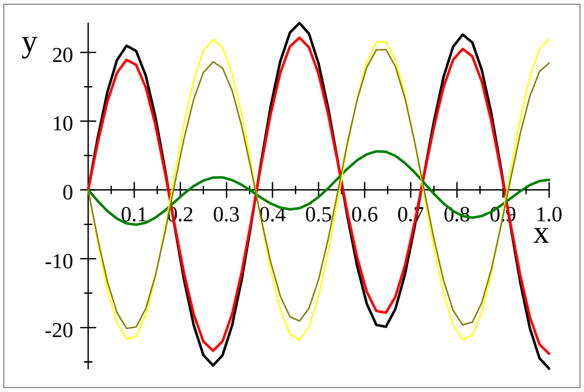
Therefore

$$D_2 = -3 \quad D_5 = 23 \quad D_n = 0 \;\; n \neq 2,5$$

The final solution is then

$$u(x,t) = -3\sin\left(\frac{5\pi x}{2}\right)\cos\left(\frac{5\pi t}{8}\right) + 23\sin\left(\frac{11\pi x}{2}\right)\cos\frac{11\pi t}{8}$$

$u(x,0) = -3\sin\frac{5}{2}\pi x + 23\sin\frac{11}{2}\pi x$



$u(x,.1) = -3\sin\frac{5}{2}\pi x\cos 0.0625\pi + 23\sin\frac{11}{2}\pi x\cos 0.1375\pi$
$u(x,.4) = -3\sin\frac{5}{2}\pi x\cos 0.25\pi + 23\sin\frac{11}{2}\pi x\cos 0.55\pi$
$u(x,.6) = -3\sin\frac{5}{2}\pi x\cos 0.375\pi + 23\sin\frac{11}{2}\pi x\cos 0.825\pi$
$u(x,.8) = 23\sin\frac{11}{2}\pi x\cos 1.1\pi$

Example Solve

$$\text{PDE} \quad u_{xx} - 16u_{tt} = 0$$
$$\text{BCs} \quad u(0,t) = 0 \qquad u_x(1,t) = 0$$
$$\text{IC} \quad u(x,0) = -6\sin\left(\frac{3\pi x}{2}\right) + 13\sin\left(\frac{11\pi x}{2}\right)$$
$$\text{IC} \quad u_t(x,0) = 0$$

You must derive the solution. Your solution should not have any arbitrary constants in it. Show **all** steps.
Solution: Let $u(x,t) = X(x)T(t)$. Then the PDE implies

$$X''T = 16XT''$$

or

$$\frac{X''}{X} = 16\frac{T''}{T} = -\lambda^2$$

since we will need sines and cosines in the X part of the solution.

Thus

$$X'' + \lambda^2 X = 0$$

$$T'' + \frac{\lambda^2}{16}T = 0$$

The BCs are

$$X(0) = X'(1) = 0$$

$$X(x) = a_n \sin\lambda x + b_n \cos\lambda x$$

$X(0) = 0$ implies that $b_n = 0,$ so

$$X(x) = a_n \sin\lambda x$$

$$X'(x) = a_n\lambda\cos\lambda x$$

so

$$X'(1) = a_n\lambda\cos\lambda = 0$$

Hence $\lambda = \frac{2n+1}{2}\pi, \;\; n = 0,1,2,\ldots$ and

$$X_n(x) = A_n \sin\left(\frac{2n+1}{2}\right)\pi x \quad n = 0,1,2,\ldots$$

Also

$$T'' + \frac{\lambda^2}{16}T = T'' + \frac{(2n+1)^2\pi^2}{64}T = 0$$

$$T_n(t) = c_n \sin\left(\frac{2n+1}{8}\right)\pi t + d_n\cos\left(\frac{2n+1}{8}\right)\pi t$$

$u_t(x,0) = 0$ implies that $c_n = 0$ and

$$T_n(t) = d_n\cos\left(\frac{2n+1}{8}\right)\pi t$$

Thus

$$u_n(x.t) = B_n \sin\left(\frac{2n+1}{2}\right)\pi x\cos\left(\frac{2n+1}{8}\right)\pi t \quad n = 0,1,2,\ldots$$

Let

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x.t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2}\right)\pi x\cos\left(\frac{2n+1}{8}\right)\pi t$$

$$u(x,0) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2}\right)\pi x = -6\sin\left(\frac{3\pi x}{2}\right) + 13\sin\left(\frac{11\pi x}{2}\right)$$

Therefore $B_1 = -6, B_5 = 13$ and $B_n = 0$ for $n \neq 1,5$ so

$$u(x,t) = -6\sin\left(\frac{3\pi x}{2}\right)\cos\left(\frac{3\pi}{8}\right)t + 13\sin\left(\frac{11\pi x}{2}\right)\cos\left(\frac{11\pi}{8}\right)t$$

Example Solve:

P.D.E.: $u_{xx} - 16u_{tt} = 0$

B.C.'s: $u(0,t) = 0 \qquad u_x(1,t) = 0$

I.C.: $u(x,0) = -3\sin\frac{5\pi x}{2} + 23\sin\frac{11\pi x}{2}; \quad u_t(x,0) = 2\pi\sin\frac{3\pi x}{2}$

Solution: We assume

$$u(x,t) = X(x)T(t)$$

The PDE implies

$$\frac{X''}{X} = 16\frac{T''}{T} = k \quad k \text{ a constant}$$

Then we have the two ordinary DEs

$$X'' - kX = 0$$

$$T'' - \frac{1}{16}kT = 0$$

The boundary conditions for $X(x)$ are

$$X(0) = X'(1) = 0$$

so that the eigenvalue problem for X is

$$X'' - kX = 0 \quad X(0) = X'(1) = 0$$

For nontrivial solutions we let $k = -\beta^2, \beta \neq 0$ and get

$$X'' + \beta^2 X = 0$$

so

$$X(x) = C_1 \sin \beta x + C_2 \cos \beta x$$

$$X(0) = 0 \Rightarrow C_2 = 0$$

Thus

$$X'(x) = C_1 \beta \cos \beta x$$

and $X'(1) = 0 \Rightarrow$

$$\beta = \left(\frac{2n+1}{2}\right)\pi \quad n = 0, 1, 2, \ldots$$

Therefore

$$X_n(x) = a_n \sin\left(\frac{2n+1}{2}\right)\pi x \quad n = 0, 1, 2, \ldots$$

Since

$$k = -\beta^2 = \left(\frac{2n+1}{2}\right)^2 \pi^2$$

The equation for $T(t)$ becomes

$$T'' + \frac{1}{16}\left(\frac{2n+1}{2}\right)^2 \pi^2 T = 0$$

so

$$T_n(t) = b_n \sin\left(\frac{2n+1}{8}\right)\pi t + c_n \cos\left(\frac{2n+1}{8}\right)\pi t \quad n = 0, 1, 2, \ldots$$

Thus

$$u_n(x,t) = X_n(x)T_n(t) = D_n \sin\left(\frac{2n+1}{2}\right)\pi x \sin\left(\frac{2n+1}{8}\right)\pi t + E_n \sin\left(\frac{2n+1}{2}\right)\pi x \cos\left(\frac{2n+1}{8}\right)\pi t \quad n = 0, 1, 2, \ldots$$

Let

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty}\left[D_n \sin\left(\frac{2n+1}{2}\right)\pi x \sin\left(\frac{2n+1}{8}\right)\pi t + E_n \sin\left(\frac{2n+1}{2}\right)\pi x \cos\left(\frac{2n+1}{8}\right)\pi t\right]$$

$$u_t(x,t) = \sum_{n=0}^{\infty}\left[D_n\left(\frac{2n+1}{8}\right)\pi \sin\left(\frac{2n+1}{2}\right)\pi x \cos\left(\frac{2n+1}{8}\right)\pi t - E_n\left(\frac{2n+1}{8}\right)\pi \sin\left(\frac{2n+1}{2}\right)\pi x \sin\left(\frac{2n+1}{8}\right)\pi t\right]$$

Then

$$u(x,0) = \sum_{n=0}^{\infty} E_n \sin\left(\frac{2n+1}{2}\right)\pi x = -3\sin\frac{5\pi x}{2} + 23\sin\frac{11\pi x}{2}$$

Therefore

$$E_2 = -3 \quad E_5 = 23 \quad E_n = 0 \;\; n \neq 2,5$$

$$u_t(x.0) = \sum_{n=0}^{\infty} D_n\left(\frac{2n+1}{8}\right)\pi \sin\left(\frac{2n+1}{2}\right)\pi x = 2\pi \sin\frac{3\pi x}{2}$$

Thus $D_1\left(\frac{3}{8}\right)\pi = 2\pi$ so $D_1 = \frac{16}{3}$ and $D_n = 0 \;\; n \neq 1$

The final solution is then

$$u(x,t) = \frac{16}{3}\sin\left(\frac{3\pi x}{2}\right)\cos\frac{3\pi t}{8} - 3\sin\left(\frac{5\pi x}{2}\right)\cos\left(\frac{5\pi t}{8}\right) + 23\sin\left(\frac{11\pi x}{2}\right)\cos\frac{11\pi t}{8}$$

The Heat Equation

Consider a cylinder parallel to x –axis

Let u denote the temperature in the cylinder. Suppose the ends $x = 0$ and $x = L$ are kept at zero temperature whereas at $t = 0$ the initial temperature distribution is $u = f(x)$. It may be shown that $u = u(x,t)$ satisfies the P.D.E.

$$u_{xx} = \frac{1}{k}u_t \qquad 0 < x < L, \qquad t > 0, \qquad (1)$$

where k is a constant and $k > 0$
.
Equation (1) is called the heat equation. The physical conditions of the problem imply

$$B.C. \;\; u(0,t) = 0 = u(L,t) \qquad t \geq 0 \qquad (2)$$

$$I.C. \;\; u(x,0) = f(x) \qquad 0 \leq x \leq L \qquad (3)$$

We want to determine $u(x,t)$, i.e. the temperature in the cylinder at any point x at any time t. Again we use separation of variables. The assumption $u(x,t) = X(x)T(t)$ leads to

$$\frac{X''(x)}{X(x)} = \frac{1}{k}\frac{T'(t)}{T(t)} = -\lambda^2$$

$\Rightarrow X'' + \lambda^2 X = 0 \qquad X(0) = X(L) = 0$ and $T' + k\lambda^2 T = 0.$

$\Rightarrow X_n = c_n \sin\frac{n\pi x}{L} \qquad n = 1, 2, \ldots \qquad \lambda_n = \frac{n\pi}{L} \Rightarrow$

$$T' + k\frac{n^2\pi^2}{L^2}T = 0$$

$\Rightarrow$

$$T(t) = d_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$\Rightarrow$

$$u_n(x,t) = a_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}\sin\frac{n\pi x}{L}$$

satisfies (1) and (2) $\Rightarrow$

$$u_n(x,t) = \sum_1^\infty a_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}\sin\frac{n\pi x}{L}$$

also satisfies (1) and (2).

We need to satisfy (3) namely, $u(x,0) = f(x)$ However,

$$u(x,0) = \sum_1^\infty a_n \sin\frac{n\pi x}{L}$$

Thus we take a_n to be the Fourier sine coefficients of $f(x)$. Hence

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Remark. The factor $e^{-\left(\frac{n\pi}{L}\right)^2 kt} \to 0$ as $t \to \infty \Rightarrow \lim_{t\to\infty} u(x,t) = 0$ as expected from the physical problem.

Example Solve the problem:

$$\text{P.D.E.:} \quad u_{xx} - 8u_t = 0$$
$$\text{B.C.:} \ u(0,t) = 0 \qquad u_x(1,t) = 0$$
$$\text{I.C.:} u(x,0) = -2\sin\frac{3\pi}{2}x + 10\sin\frac{9\pi}{2}x$$

Solution: Let $u(x,t) = X(x)T(t)$. Then the PDE implies

$$\frac{X''}{X} = 8\frac{T'}{T} = k \ \ k \text{ a constant}$$

Then we have the two ODEs

$$X'' - kX = 0$$
$$T' - \frac{1}{8}kT = 0$$

The BCs for $X(x)$ are

$$X(0) = X'(1) = 0$$

The boundary conditions for $X(x)$ are

$$X(0) = X'(1) = 0$$

so that the eigenvalue problem for X is

$$X'' - kX = 0 \quad X(0) = X'(1) = 0$$

For nontrivial solutions we let $k = -\beta^2, \beta \neq 0$ and get

$$X'' + \beta^2 X = 0$$

so

$$X(x) = C_1 \sin\beta x + C_2 \cos\beta x$$
$$X(0) = 0 \Rightarrow C_2 = 0$$

Thus

$$X'(x) = C_1 \beta \cos\beta x$$

and $X'(1) = 0 \Rightarrow$

$$\beta = \left(\frac{2n+1}{2}\right)\pi \quad n = 0, 1, 2, \ldots$$

Therefore

$$X_n(x) = a_n \sin\left(\frac{2n+1}{2}\right)\pi x \quad n = 0, 1, 2, \ldots$$

The equation for $T(t)$ with $k = -\beta^2 = \left(\frac{2n+1}{2}\right)^2 \pi^2$ is

$$T' + \frac{1}{8}\left(\frac{2n+1}{2}\right)^2 \pi^2 T = 0$$

Thus

$$T_n(t) = b_n e^{-\frac{1}{8}\left(\frac{2n+1}{2}\right)^2 \pi^2 t} \quad n = 0, 1, 2, \ldots$$

Therefore we have

$$u_n(x,t) = D_n \sin\left(\frac{2n+1}{2}\right)\pi x e^{-\frac{1}{8}\left(\frac{2n+1}{2}\right)^2 \pi^2 t} \quad n = 0, 1, 2, \ldots$$

To satisfy the initial condition we let

$$u(x,t) = \sum_{n=0}^{\infty} D_n \sin\left(\frac{2n+1}{2}\right)\pi x e^{-\frac{1}{8}\left(\frac{2n+1}{2}\right)^2 \pi^2 t}$$

Now

$$u(x,0) = \sum_{n=0}^{\infty} D_n \sin\left(\frac{2n+1}{2}\right)\pi x = -2\sin\frac{3\pi}{2}x + 10\sin\frac{9\pi}{2}x$$

This means

$$D_1 = -2 \quad D_4 = 10 \text{ and } D_n = 0, \quad n \neq 1, 2$$

The solution to the problem is then

$$u(x,t) = -2\sin\left(\frac{3}{2}\right)\pi x e^{-\frac{1}{8}\left(\frac{3}{2}\right)^2 \pi^2 t} + 10\sin\left(\frac{9}{2}\right)\pi x e^{-\frac{1}{8}\left(\frac{9}{2}\right)^2 \pi^2 t}$$

Additional Examples

Example Wave Equation Example

Problem 1 Section 10.6

Find a formal solution to the vibrating string problem governed by the given initial-boundary value problem.

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) &= u(1,t) = 0, \quad t > 0 \\ u(x,0) &= x(1-x), \quad 0 < x < 1 \\ u_t(x,0) &= \sin 7\pi x, \quad 0 < x < 1 \end{aligned}$$

Let $u(x,t) = X(x)T(t)$. Then the PDE leads to

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda^2$$

We then have two ODEs

$$\begin{aligned} X'' + \lambda^2 X &= 0 \\ T'' + \lambda^2 T &= 0 \end{aligned}$$

Therefore

$$X(x) = a\cos\lambda x + b\sin\lambda x$$

The BCs for $X(x)$ are $X(0) = X(1) = 0$. Thus, $a = 0$ and $\lambda = n\pi$, $n = 1,2,\ldots$ and

$$X_n(x) = c_n \sin n\pi x \quad n = 1,2,\ldots$$

Also

$$T_n(t) = d_n \cos n\pi t + e_n \sin n\pi t \quad n = 1,2,\ldots$$

so

$$u_n(x,t) = [a_n \cos n\pi t + b_n \sin n\pi t]\sin n\pi x \quad n = 1,2,\ldots$$

Thus we let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}[a_n \cos n\pi t + b_n \sin n\pi t]\sin n\pi x$$

We want

$$u(x,0) = x(1-x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$$

Therefore the constants a_n are given by the formula for the Fourier sine series coefficients with $L = 1$ so

$$\begin{aligned} a_n &= \frac{2}{1}\int_0^1 x(1-x)\sin n\pi x dx \\ &= 2\left(\int_0^1 x\sin n\pi x dx + \int_0^1 x^2 \sin n\pi x dx\right) \end{aligned}$$

Integration by parts yields

$$\int_0^1 x\sin n\pi x dx = -\frac{1}{n\pi}\cos n\pi = -\frac{(-1)^n}{n\pi}$$

and

$$\int_0^1 x^2 \sin n\pi x dx = -\frac{1}{n\pi}\cos n\pi - \frac{2}{n^2\pi^2}\left(-\frac{1}{n\pi}\cos n\pi + \frac{1}{n\pi}\right)$$
$$= -\frac{(-1)^n}{n\pi} + \frac{2[(-1)^n - 1]}{n^3\pi^3}$$

Therefore for $n = 1, 2, \ldots$

$$a_n = 2\left\{-\frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} - \frac{2[(-1)^n - 1]}{n^3\pi^3}\right\} = -\frac{4[(-1)^n - 1]}{n^3\pi^3}$$

Note that

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{n^3\pi^3} & \text{if } n \text{ is odd} \end{cases}$$

Since

$$u_t(x,t) = \sum_{n=1}^{\infty}[-a_n(n\pi)\sin n\pi t + b_n(n\pi)\cos n\pi t]\sin n\pi x$$

then

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n(n\pi)\sin n\pi x = \sin 7\pi x$$

Hence $7\pi b_7 = 1$ so $b_7 = \frac{1}{7\pi}$ and $b_n = 0$ for $n \neq 7$.

Substituting these constants into the expression

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}[a_n \cos n\pi t + b_n \sin n\pi t]\sin n\pi x$$

above and letting $n = 2k+1, \ \ k = 0, 1, 2, \ldots$ since n is odd yields

$$u(x,t) = \frac{1}{7\pi}\sin 7\pi t \sin 7\pi x + \sum_{k=0}^{\infty} \frac{8}{[(2k+1)\pi]^3}\cos(2k+1)\pi t \sin(2k+1)\pi x$$

Example Use separation of variables, $u(x,t) = X(x)T(t)$, to find two ordinary differential equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$-3x^2t^4\frac{\partial^2 u}{\partial x^2} + (x-2)^4(t+6)^3\frac{\partial^2 u}{\partial t^2} = 0.$$

Note: Do **not** solve these ordinary differential equations.

Solution: $u_x(x,t) = X'(x)T(t), u_{xx} = X''T$

$$-3x^2t^4X''(x)T(t) + (x-2)^4(t+6)^3X(x)T''(t) = 0$$

$\Rightarrow$

$$\frac{-3x^2X''}{(x-2)^4X} = -\frac{(t+6)^3T''}{t^4T} = k.$$

$\Rightarrow$

$$3x^2X'' + k(x-2)^4X = 0 \text{ and } (t+6)^3T'' + kt^4T = 0.$$

Example Solve

$$\text{PDE} \qquad u_{xx} = 4u_{tt}$$
$$\text{BCS} \quad u_x(0,t) = 0 \qquad u_x(\pi,t) = 0$$
$$\text{ICs} \quad u(x,0) = 0 \qquad u_t(x,0) = -9\cos(4x) + 16\cos(8x)$$

You must derive the solution. Your solution should not have any arbitrary constants in it.
Solution:
Let $u(x,t) = X(x)T(t)$. Then differentiating and substituting in the PDE yields

$$X''T = 4XT''$$
$$\Rightarrow$$
$$\frac{X''}{X} = 4\frac{T''}{T}$$

Using the argument that the left hand side is purely a function of x and the right hand side is purely a function of t, and the only way that they can be equal is if they are equal to a constant, we get

$$\frac{X''}{X} = 4\frac{T''}{T} = k \qquad k \text{ a constant}$$

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T'' - \frac{1}{4}kT = 0$$

The boundary condition $u_x(0,t) = 0$ implies, since $u_x(x,t) = X'(x)T(t)$ that $X'(0)T(t) = 0$. We cannot have $T(t) = 0$, since this would imply that $u(x,t) = 0$. Thus $X'(0) = 0$. Similarly, the boundary condition $u_x(\pi,t) = 0$ leads to $X'(\pi) = 0$.

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X'(0) = X'(\pi) = 0$$

For $k > 0$, the only solution is $X = 0$. For $k = 0$ we have $X = Ax + B$. $X'(x) = A$, so the BCs imply that $X'(0) = X'(\pi) = A = 0$.

$$X(x) = B, \quad B \neq 0$$

is a nontrivial solution corresponding to the eigenvalue $k = 0$.
For $k < 0$, let $-k = \alpha^2$, where $\alpha \neq 0$. Then we have the equation

$$X'' + \alpha^2 X = 0$$

and

$$X(x) = c_1 \sin\alpha x + c_2 \cos\alpha x$$
$$X'(x) = c_1\alpha\cos\alpha x - c_2\alpha\sin\alpha x$$
$$X'(0) = c_1\alpha = 0$$

so $c_1 = 0$.

$$X'(\pi) = -c_2\alpha\sin\alpha\pi = 0$$

Therefore $\alpha = n,\ n = 1, 2, \ldots$ and the solution is

$$k = -n^2 \qquad X_n(x) = a_n \cos nx \qquad n = 1, 2, 3, \ldots$$

The case $k = 0$ implies that the equation for T becomes $T'' = 0,$ so $T = At + B.$ The initial condition $u(x, 0) = 0$ implies $X(x)T(0) = 0$ so that $T(0) = 0.$ Thus $B = 0$ and $T = At$ for $k = 0.$
Substituting the values of $k = -n^2$ into the equation for $T(t)$ leads to

$$T'' + \frac{n^2}{4} T = 0$$

which has the solution

$$T_n(t) = B_n \sin \frac{nt}{2} + C_n \cos \frac{nt}{2}, \quad n = 1, 2, 3, \ldots$$

The initial condition $u(x, 0) = 0$ implies $X(x)T(0) = 0$ so that $T(0) = 0.$ Thus $c_n = 0.$ For $n = 0$ the equation for T becomes $T'' = 0,$ and has the solution $T(t) = B_0 t + C_0.$ The condition $T(0) = 0$ implies that $C_0 = 0,$ so $T_0(t) = B_0 t$
We now have the solutions

$$u_n(x, t) = X_n(x) T_n(t) = A_n \cos nx \sin \frac{nt}{2} \qquad n = 1, 2, 3, \ldots$$
$$u_0(x, t) = A_0 t$$

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = A_0 t + \sum_{n=1}^{\infty} A_n \cos nx \sin \frac{nt}{2}$$

satisfies the PDE, the boundary conditions, and the first initial condition. Since

$$u_t(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{n}{2} \right) \cos nx \cos \frac{nt}{2}$$

the last initial condition leads to

$$u_t(x, 0) = -9 \cos(4x) + 16 \cos(8x) = A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{n}{2} \right) \cos nx.$$

Matching the cosine terms on both sides of this equation leads to

$A_4 \left(\frac{4}{2} \right) = -9$ so that $A_4 = -\frac{9}{2}$ and $A_8 \left(\frac{8}{2} \right) = 16$ so that $A_8 = 4.$ All of the other constants must be zero, since there are no cosine terms or constant terms on the left to match with. Thus

$$u(x, t) = -\frac{9}{2} \cos 4x \sin 2t + 4 \cos 8x \sin 4t$$

Example Consider the non-homogeneous problem

$$\text{P.D.E.}: \; u_{xx} = 9u_t$$
$$\text{B.C.'s}: \; u_x(0,t) = 0 \qquad u(1,t) = 2$$
$$\text{I.C.}: \; u(x,0) = -3\cos\frac{7\pi}{2}x + 2$$

i)
Let

$$v(x,t) = u(x,t) - 2$$

and show that $v(x,t)$ satisfies the homogeneous problem

$$\text{P.D.E.}: \; v_{xx} = 9v_t$$
$$\text{B.C.}: \; v_x(0,t) = 0 \qquad v(1,t) = 0$$
$$\text{I.C.}: \; v(x,0) = -3\cos\frac{7\pi}{2}x$$

Solution to i)

$$u_{xx}(x,t) = v_{xx}(x,t) \qquad u_x(x,t) = v_x(x,t)$$

$$u_{tt}(x,t) = v_{tt}(x,t) \qquad u_t(x,t) = v_t(x,t)$$

$$u(1,t) = 2 \text{ and } u(x,t) - 2 = v(x,t) \Rightarrow v(1,t) = 0$$

$$u_x(0,t) = 0 \Rightarrow v_x(0,t) = 0$$

$$u(x,0) = -3\cos\frac{7\pi}{2} + 2 \text{ and } u(x,t) - 2 = v(x,t) \Rightarrow v(x,0) = -3\cos\frac{7\pi}{2}$$

ii)
Solve the above problem for $v(x,t)$.
Solution to ii) Let $v(x,t) = X(x)T(t)$
then

$$X''T = 9XT' \Rightarrow \frac{X''}{X} = 9\frac{T'}{T} = k$$

resulting in the ordinary differential equations:

$$X'' - kX = 0 \text{ and } T' - \frac{k}{9}T = 0$$

Boundary Conditions become:

$$X'(0)T(t) = 0 \text{ and } X(1)T(t) = 0$$
$$\Rightarrow X'(0) = 0 \text{ and } X(1) = 1$$

Solving the differential equation $X'' - kX = 0$ consider all values of k
$k < 0$ let $k = -u^2; \quad u > 0$

$$X'' + u^2 X = 0$$

has the solution:

$$X(x) = c_1 \cos ux + c_2 \sin ux$$

and

$$X'(x) = -c_1 u \sin ux + c_2 u \cos ux$$

B.C. $\Rightarrow X(1) = c_1 \cos u + c_2 \sin u = 0$ and $X'(0) = c_2 u = 0$

$\Rightarrow c_2 = 0$ thus $c_1 \cos u = 0$

$$\Rightarrow u_n = \frac{(2n-1)\pi}{2} \quad n = 1, 2, \ldots$$

$$\Rightarrow k_n = -\frac{(2n-1)^2\pi^2}{4} \quad n = 1, 2, \ldots$$

so

$$X_n(x) = c_n \cos \frac{(2n-1)\pi}{2} x \quad n = 1, 2, \ldots$$

The other cases for k, namely $k = 0$ and $k > 0$ yield only the trivial solution since

$k = 0 \Rightarrow X'' = 0$ which has the solution: $X(x) = c_1 x + c_2$ and $X'(x) = c_1$

B.C. $\Rightarrow X(1) = c_1 + c_2 = 0$ and $X'(0) = c_1 = 0 \Rightarrow c_2 = 0$

thus $X(x) \equiv 0$ is the trivial solution.

$k > 0$ let $k = u^2$; $u > 0$

$X'' - u^2 X = 0$ has the solution: $X(x) = c_1 e^{ux} + c_2 e^{-ux}$

and $X'(x) = c_1 u e^{ux} - c_2 u e^{-ux}$

B.C. $\Rightarrow X'(0) = c_1 u - c_2 u = 0 \Rightarrow c_1 = c_2$

and $X(1) = c_1 e^{u} + c_2 e^{-u} = 0 \Rightarrow c_1 e^{u} + c_1 e^{-u} = 0 \Rightarrow c_1(e^{u} + e^{-u}) = 0$

$\Rightarrow c_1 = c_2 = 0$ thus $X(x) \equiv 0$ is the trivial solution.

Using the non-trivial solution

$$k_n = -\frac{(2n-1)^2\pi^2}{4} \qquad X_n(x) = c_n \cos \frac{(2n-1)\pi}{2} x, \quad n = 1, 2, \ldots$$

the equation

$$T' - \frac{k}{9} T = 0$$

becomes

$$T' + \frac{(2n-1)^2\pi^2}{36} T = 0$$

solving by separating

$$\frac{T'}{T} = -\frac{(2n-1)^2\pi^2}{36} \Rightarrow \int \frac{T'}{T} = -\int \frac{(2n-1)^2\pi^2}{36}$$

$$\Rightarrow \ln T = -\frac{(2n-1)^2\pi^2}{36} t + c \Rightarrow T_n(t) = c_n e^{-\frac{(2n-1)^2\pi^2}{36} t}$$

Therefore

$$\begin{aligned} v_n(x,t) &= X_n(x) T_n(t) \\ &= c_n \cos \frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2\pi^2}{36} t} \end{aligned}$$

so we let

$$v(x,t) = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2\pi^2}{36}t}$$

Using I.C. to compute coefficients we have

$$v(x,0) = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2} = -3\cos\frac{7\pi x}{2}$$

by equating coefficients: $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = -3, c_4 = 0, \ldots$

$$v(x,t) = -3\cos\frac{7\pi x}{2} e^{-\frac{49\pi^2}{36}t}$$

is the solution.

iii) Now use the results of b) i) and ii) to find $u(x,t)$.

Solution to iii)

$$u(x,t) = v(x.t) + 2$$

so

$$u(x,t) = -3\cos\frac{7\pi x}{2} e^{-\frac{49\pi^2}{36}t} + 2$$