Ma 221

Chapter 2 - Special Methods for First Order Equations

Consider the equation

$$M(x,y) + N(x,y)y' = 0 \quad (1)$$

This equation is first order and first degree. The functions $M(x,y)$ and $N(x,y)$ are given.
Often we write this as

$$M(x,y)dx + N(x,y)dy = 0 \quad (2)$$

Separation of Variables

Equation (2) takes a simple form in the special case when

$$M(x,y) = A(x) \text{ and } N(x,y) = B(y).$$

$\Rightarrow$

$$A(x)dx + B(y)dy = 0$$

That is the variables separate.
If we Integrate $\Rightarrow$

$$\int A(x)dx + \int B(y)dy = c.$$

Example $x^2dx + ydy = 0 \Rightarrow$

$$\int x^2dx + \int ydy = c.$$

Which leads to

$$\frac{x^3}{3} + \frac{y^2}{2} = c.$$

Now consider the I.V.P.

$$\text{D.E. } A(x)dx + B(y)dy = 0$$

$$\text{I.C. } y(x_0) = y_0$$

Integrating from $(x_0, y_0) \to (x,y) \Rightarrow$

$$\int_{x_0}^{x} A(x)dx + \int_{y_0}^{y} B(y)dy = 0$$

.

Example D.E. $\cos x \ dx + y^2 dy = 0$ I.C. $y(\pi) = 0$

$$\int_\pi^x \cos x \ dx + \int_0^y y^2 \ dy = 0 \quad \Rightarrow \quad \sin x \,\Big|_\pi^x + \frac{y^3}{3} \,\Big|_0^y = 0$$

or $\sin x - \sin \pi + \frac{y^3}{3} = 0 \quad \Rightarrow \sin x + \frac{y^3}{3} = 0 \quad \Rightarrow$

$$y^3 = -3 \sin x$$

Example Solve $xdy + ydx = 0$ This equation is not separable as is.
Divide by $xy \Rightarrow$

$$\frac{dy}{y} + \frac{dx}{x} = 0$$

$\Rightarrow \ln x + \ln y = c$ or $\ln|xy| = c \Rightarrow | \ xy \ | = k$
$\Rightarrow \ xy = \pm k \quad \Rightarrow$

$$y = \frac{k}{x} \qquad \forall x \neq 0.$$

Example Solve

$$(2y - \sin y)y' + t = \sin t \quad y(0) = 1$$

Solution: We rewrite the equation as

$$(2y - \sin y)dy + (t - \sin t)dt = 0$$

which is separable. Integrating we have

$$y^2 + \cos y + \frac{t^2}{2} + \cos t = c$$

The initial condition implies

$$1 + \cos 1 + 1 = c$$

so

$$y^2 + \cos y + \frac{t^2}{2} + \cos t = 2 + \cos 1$$

Example This example is a video slide show. Slide Example

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First Order linear differential equations

Clearly not all equations are as simple as the equation $A(x)dx + B(y)dy = 0$. Consider the equation

$$a(x)\frac{dy}{dx} + b(x)y = c(x).$$

Assuming $a(x) \neq 0$ we divide by $a(x) \quad \Rightarrow$

$$y' + P(x)y = Q(x) \quad (1)$$

or

$$dy + (P(x)y - Q(x))dx = 0.$$

We want to solve (1). Consider first the homogeneous problem

$$y' + P(x)y = 0.$$

$\Rightarrow$

$$\frac{dy}{y} + P(x)dx = 0$$

which is separable.

$\Rightarrow \ln|y| + \int P(x)dx = c \qquad \Rightarrow |\ y\ | = e^{c-\int P(x)dx}$

Hence

$$y = \pm e^c e^{-\int P(x)dx} = ke^{-\int P(x)dx}$$

is the homogeneous solution.

Non-homogeneous case:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

We shall use variation of parameters. Note that any constant times $e^{-\int P(x)dx}$ is also a solution of the homogeneous equation (1).

To solve the nonhomogeneous equation we shall try a function times $e^{-\int P(x)dx}$ i.e.

$$y = v(x)e^{-\int P(x)dx}$$

$\Rightarrow$

$$y' = v'e^{-\int P(x)dx} + ve^{-\int P(x)dx}(-P(x)).$$

Now the D.E. $\Rightarrow$

$$v'e^{-\int P(x)dx} + ve^{-\int P(x)dx}(-P(x)) + P(x)ve^{-\int P(x)dx} = Q$$

$\Rightarrow$

$$v' = Qe^{\int P(x)dx}$$

$\Rightarrow$

$$v = \int Qe^{\int P(x)dx} + c.$$

Therefore the solution is

$$y = ve^{-\int P(x)dx} = ce^{-\int P(x)dx} \qquad + \left(\int Qe^{\int P(x)dx}\right) e^{-\int P(x))}$$

homogeneous solution + particular solution

Example: $y' + \frac{y}{x+1} = x^2 \qquad P = \frac{1}{x+1} \;\; Q = x^2$.

Consider $y' + \frac{y}{x+1} = 0 \Rightarrow \frac{dy}{y} + \frac{dx}{x+1} = 0$ or $\ln|y(x+1)| = c \Rightarrow y = k/(x+1)$.

Using the formula for the homogeneous solution, we have

$$y = ke^{-\int P(x)dx} = ke^{-\int \frac{dx}{x+1}} = ke^{-ln(x+1)} = \frac{k}{x+1}$$

.

We now solve the nonhomogeneous equation. Since $y = ve^{-\int P(x)dx} = \frac{v}{x+1}$

⇒

$$y' = \frac{v'}{x+1} - \frac{v}{(x+1)^2}$$

.

The D.E. ⇒

$$\frac{v'}{x+1} - \frac{v}{(x+1)^2} + \frac{v}{(x+1)^2} = x^2$$

⇒

$$v' = x^2(x+1) = x^3 + x^2$$

Thus

$$v = \frac{x^4}{4} + \frac{x^3}{3} + c$$

and therefore

$$y = \frac{c}{x+1} + \frac{\frac{x^4}{4} + \frac{x^3}{3}}{x+1}$$

.

Remark: The variation of parameters method works because the assumption $y = ve^{-\int Pdx}$ leads to $v' = Qe^{\int Pdx}$. Since $v = y\ e^{\int Pdx}$ ⇒

$$\frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx}$$

⇒

$$e^{\int Pdx}[y' + Py] = Qe^{\int Pdx}.$$

Therefore if we multiply the original equation by $e^{\int Pdx}$ ⇒ we get an integrable form right away.

Example $y' + \frac{y}{x+1} = x^2$ (Again)

$P = \frac{1}{x+1}$ $\quad e^{\int Pdx} = e^{\int \frac{dx}{x+1}} = e^{\ln(x+1)} = x + 1$ ⇒

$$(x+1)y' + y = x^2(x+1)$$

or

$$\frac{d}{dx}[(x+1)y] = x^2(x+1)$$

⇒

$$(x+1)y = \frac{x^4}{4} + \frac{x^3}{3} + c$$

as before.

Summary:

To solve $y' + Py = Q$ multiply both sides by the integrating factor $I = e^{\int P dx}$. Then the L.H.S. becomes

$$\frac{d}{dx}\left(ye^{\int P dx}\right) = e^{\int P dx} Q$$

and the solution is found by integrating both sides. This is called the Method of the Integrating Factor.

We can use the above to solve the I.V.P.

$$\text{D.E. } y' + P(x)y = Q(x)$$
$$\text{I.C. } y(x_0) = y_0$$

Use the integrating factor

$$I = e^{\int_{x_0}^{x} P(t)dt}$$

and integrate both sides from x_0 to x.

Example Solve

$$ty' + 4y = 6t^2 \quad y(1) = 3 \quad t > 0$$

Solution: This equation is first order linear and may be written as

$$y + \frac{4}{t}y' = 6t$$

We multiply the DE by $e^{\int P(t)dt} = e^{\int \frac{4}{t}dt} = e^{4\ln t} = t^4$ and get

$$t^4 y' + 4t^3 y = 6t^5$$

or

$$\frac{d}{dt}(t^4 y) = 6t^5$$

Hence

$$t^4 y = t^6 + c$$

and

$$y = t^2 + \frac{c}{t^4}$$

The initial condition yields

$$3 = 1 + c \text{ or } c = 2$$

so

$$y = t^2 + \frac{2}{t^4}$$

Example Solve

$$y' = 2t^{-1} + e^{-y} \quad y(1) = 0$$

Solution: Rewrite the equation as

$$e^y y' - \frac{2}{t} e^y = 1$$

Let $z = e^y$. Then $z' = e^y y'$ and the DE becomes

$$z' - \frac{2}{t} z = 1$$

This is first order linear in z. Multiply the DE by $e^{-\int \frac{2}{t} dt} = e^{-2\ln t} = t^{-2}$ to get

$$t^{-2} z - 2t^{-3} z = t^{-2}$$

or

$$(t^{-2} z)' = t^{-2}$$

Hence

$$t^{-2} z = -t^{-1} + c$$

or

$$z = e^y = -t + ct^2$$

The initial condition implies

$$1 = -1 + c \text{ or } c = 2$$

so

$$e^y = -t + 2t^2$$

or

$$y = \ln(2t^2 - t)$$

Example Here are two video slide show examples. Slide Example 1 Slide Example 2

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Example The equation

$$\frac{dy}{dx} + P(x) y = Q(x) y^n \qquad n \text{ any real number}$$

is known as Bernoulli's equation.

We shall suppose that $n \neq 0$ or 1, since we already know how to solve the equation for these two cases. Multiplying by y^{-n} yields

$$y^{-n} y' + P(x) y^{-n+1} = Q(x)$$

Let $z = y^{-n+1}$ Then $z' = (-n+1) y^{-n} y' \Rightarrow$

$$\frac{z'}{1-n} + P(x) z = Q(x).$$

This is a linear differential equation for z which can be solved. For example, consider the equation

$$x \frac{dy}{dx} + y = xy^{-4}$$

$\Rightarrow$

$$y' + \frac{1}{x}y = y^{-4} \quad (n = -4)$$

$\Rightarrow$

$$y^4y' + \frac{y^5}{x} = 1$$

Let $z = y^5 \quad \Rightarrow z' = 5y^4y' \Rightarrow$

$$\frac{z'}{5} + \frac{z}{x} = 1$$

$\Rightarrow$

$$z' + \frac{5}{x}z = 5$$

Thus the integrating factor is $e^{\int \frac{5}{x}dx} = e^{5lnx} = x^5$ so we have

$$\frac{d}{dx}(x^5z) = 5x^5$$

$\Rightarrow$

$$x^5z = 5\frac{x^6}{6} + c$$

$\Rightarrow$

$$z = 5\frac{x}{6} + cx^{-5}$$

Since $z = y^5 \Rightarrow$

$$y^5 = 5\frac{x}{6} + cx^{-5}.$$

Example Solve

$$y' + xy = xe^{-x^2}y^{-3}$$

This is a Bernoulli equation. Multiply both sides by y^3 to get

$$y^3y' + xy^4 = xe^{-x^2}$$

Let $z = y^4$ so that $z' = 4y^3y'$. The DE may then be written as

$$\frac{z'}{4} + xz = xe^{-x^2}$$

or

$$z' + 4xz = 4xe^{-x^2}$$

This equation is a first order linear DE in z. Then $I = e^{\int Pdx} = e^{\int 4xdx} = e^{2x^2}$. Multiplying the DE by this integrating factor yields

$$z'e^{2x^2} + 4xe^{2x^2} = 4xe^{x^2}$$

or

$$\frac{d(ze^{2x^2})}{dx} = 4xe^{x^2}$$

Integrating we have

$$ze^{2x^2} = 2e^{x^2} + C$$

Since $z = y^4$ the solution is

$$y^4 = 2e^{-x^2} + Ce^{-2x^2}$$

Example Solve

$$x^2y' + xy = -y^{-\frac{3}{2}}$$

Solution: This is a Bernoulli Equation.

$$y' + \frac{1}{x}y = -\frac{1}{x^2}y^{-\frac{3}{2}} \quad \Rightarrow y^{\frac{3}{2}}\frac{dy}{dx} + \frac{1}{x}y^{\frac{5}{2}} = -\frac{1}{x^2}$$

.

Now we let

$$v = y^{1-n} = y^{1-(-\frac{3}{2})} = y^{\frac{5}{2}}.$$

Then

$$\frac{dv}{dx} = \frac{5}{2}y^{\frac{3}{2}}\frac{dy}{dx} \Rightarrow \frac{2}{5}\frac{dv}{dx} = y^{\frac{3}{2}}\frac{dy}{dx}.$$

Substituting, the equation becomes

$$\frac{2}{5}\frac{dv}{dx} + \frac{1}{x}v = -x^{-2}$$

$\Rightarrow$

$$\frac{dv}{dx} + \frac{5}{2x}v = -\frac{5}{2}x^{-2}.$$

This is a linear equation in *v*. The integrating factor is

$$I = e^{\int Pdx} = e^{\int \frac{5}{2x}dx} = e^{\frac{5}{2}\ln x} = x^{\frac{5}{2}}.$$

Multiplying the DE by *I* gives

$$\frac{dv}{dx}x^{\frac{5}{2}} + \frac{5}{2x}vx^{\frac{5}{2}} = -\frac{5}{2}x^{-2}x^{\frac{5}{2}}.$$

or

$$\frac{d}{dx}(x^{\frac{5}{2}}v) = -\frac{5}{2}x^{\frac{1}{2}}$$

$$\Rightarrow \int \frac{d}{dx}(x^{\frac{5}{2}}v)dx = \int -\frac{5}{2}x^{\frac{1}{2}}dx$$

$$\Rightarrow x^{\frac{5}{2}}v = -\frac{5}{3}x^{\frac{3}{2}} + C$$

$$\Rightarrow v = -\frac{5}{3}x^{-1} + Cx^{\frac{-5}{2}} = y^{\frac{5}{2}}$$

$$\Rightarrow y(x) = \left(-\frac{5}{3}x^{-1} + Cx^{\frac{-5}{2}}\right)^{\frac{2}{5}}$$

Exact Differential Equations

Definition: The differential expression

$$M(x,y)dx + N(x,y)dy$$

is called exact $\Leftrightarrow \exists$ a function $f(x,y)$ that is differentiable in some region R of the x,y-plane, i.e. $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist and are continuous in R and such that

$$\frac{\partial f}{\partial x} = M \qquad \frac{\partial f}{\partial y} = N \qquad \forall \ (x,y) \in R.$$

Remark: Since $df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \Rightarrow Mdx + Ndy$ is exact $\Leftrightarrow df(x,y) = Mdx + Ndy$.

Definition: The differential equation

$$M(x,y)dx + N(x,y)dy = 0 \quad (1)$$

is called an exact differential equation if the left hand side is an exact differential.

Remark: When the differential equation (1) is exact

$\Rightarrow$

$$df(x,y) = Mdx + Ndy = 0 \qquad (2).$$

Using this we may solve the differential equation. For if $y(x)$ is the solution, then (2) may be integrated with respect to x to yield

$$f(x,y) = c \qquad (3).$$

Conversely if (3) defines y as a differential function of x, then this $y(x)$ is a solution of the differential equation. For (3) $\Rightarrow$

$$\frac{df}{dx} = 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

by the chain rule. $\Rightarrow$

$$M + N\frac{dy}{dx} = 0.$$

Example: $xdy + ydx = 0$ Here $M = y$ and $N = x$

Consider $f(x,y) = xy$

$$df(x,y) = xdy + ydx$$

Since $f_x = y$ and $f_y = x$

$$df = f_x \ dx + f_y \ dy = ydx + xdy = 0$$

Therefore

$$f(x,y) = xy = c$$

determines the solution. $\Rightarrow$

$$y = \frac{c}{x}$$

Check $\frac{dy}{dx} = -\frac{c}{x^2}$ $dy = -\frac{c}{x^2}\, dx$. $\Rightarrow xdy + ydx = x\,(-\frac{c}{x^2}dx) + \frac{c}{x}dx = 0.$
Thus if we know that a certain differential equation is exact we can solve it.

Question: When is a differential equation exact? The answer is given by following theorem.

Theorem If $M(x,y)$ and $N(x,y)$ are continuous functions and have continuous partial derivatives in some region R of the x,y-plane, then the expression

$$M(x,y)dx + N(x,y)dy$$

is an exact differential $\Leftrightarrow$

$$M_y = N_x$$

throughout R.

Remark: If $f_x = M$ and $f_y = N$, then $\Rightarrow f_{xy} = M_y = N_x = f_{yx}$.

Example: $ydx + xdy$ Here $M = y$ and $N = x$ so that $M_y = 0 = N_x$ and we see that this equation is exact.

Example: $e^x \cos y dx = e^x \sin y dy$
We rewrite the equation as

$$e^x \cos y dx - e^x \sin y dy = 0.$$

Thus
$M = e^x \cos y$ and $N = -e^x \sin y$ and therefore $M_y = -e^x \sin y$ and $N_x = -e^x \sin y$. Therefore this equation is exact. $\Rightarrow \exists\, f(x,y)$ such that $f_x = M$ $f_y = N$, i.e.,

$$\frac{\partial f}{\partial x} = e^x \cos y$$

$\Rightarrow$

$$f(x,y) = \int e^x \cos y \; dx + g(y) = e^x \cos y + g(y).$$

$g(y) = ?$ We must have

$$\frac{\partial f}{\partial y} = N = -e^x \sin y.$$

Now

$$\frac{\partial f}{\partial y} = -e^x \sin y + g'(y) = -e^x \sin y$$

$\Rightarrow g'(y) = 0 \Rightarrow g = const = c$ Therefore

$$f(x,y) = e^x \cos y + c$$

$\Rightarrow$ solution is $f(x,y) = k$, i.e.

$$e^x \cos y = c + k = k'.$$

Example Solve

$$1 + y^2 + 2(t+1)y\frac{dy}{dt} = 0, \;\; y(0) = 1$$

Solution: We write the equation as

$$(1 + y^2)dt + 2(t + 1)ydy = 0$$

The $M = 1 + y^2$ and $N = 2(t + 1)y$ and

$$M_y = 2y = N_t$$

Hence the equation is exact and there exists a function $f(t,y)$ such that

$$f_t = M \text{ and } f_y = N$$

So

$$f_t = 1 + y^2 \quad \Rightarrow \quad f = t + ty^2 + g(y)$$

Also

$$f_y = 2ty + g'(y) = N = 2ty + 2y$$

Therefore

$$g(y) = y^2 + C$$

and

$$f = t + ty^2 + y^2 + C$$

and the solution is given by

$$t + ty^2 + y^2 = k$$

$y(0) = 1$ implies

$$1 = k$$

Thus

$$t + ty^2 + y^2 = 1$$

or

$$y = \sqrt{\frac{1 - t}{1 + t}}$$

Example Here is a video slide show example. Slide Example

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Integrating factors

Recall: $M(x,y)dx + N(x,y)dy = 0$ is exact $\Leftrightarrow M_y = N_x$. When the equation is exact, $\Rightarrow \exists\, f(x,y)$ such that $f_x = M$ and $f_y = N$ and $df = Mdx + Ndy = 0$. $\Rightarrow f(x,y) = c$ gives the solution to the equation.

Clearly not every differential equation is exact.

Question: Can we make $Mdx + Ndy = 0$ exact when it is not?
We want to find a function $u(x,y)$ such that when we multiply the differential equation by $u(x,y)$, then

$$uMdx + uNdy = 0$$

is exact. $u(x,y)$ is called an integrating factor.

Example $ydx + (y^2 - x)dy = 0$

$M = y \qquad N = y^2 - x$

$M_y = 1 \quad N_x = -1$ Thus the equation is not exact.
We multiply by a function u so that

$$uydx + u(y^2 - x)dy = 0$$

is exact.
$\Rightarrow$

$$[uy]_y = [u(y^2 - x)]_x$$

$\Rightarrow$

$$u_y y + u = u_x \ (y^2 - x) - u.$$

This last equation is harder to solve in general than the original. However, we do not need the general solution. We need any u which when multiplied times the equation makes it exact. If we assume $u_x = 0$, then $u = u(y)$, i.e. u is only a function of y and our partial differential equation for u becomes the ordinary differential equation

$$y\frac{du}{dy} + 2u = 0$$

which has the solution $u = \frac{1}{y^2}$. Multiplying the original equation by this u yields

$$\frac{1}{y} \ dx + (1 - \frac{x}{y^2}) \ dy = 0$$

Since now $\mathrm{M} = \frac{1}{y}$ and $\mathrm{N} = (1 - \frac{x}{y^2}) \ \Rightarrow \mathrm{M}_y = -\frac{1}{y^2} = \mathrm{N}_x \ \Rightarrow$ this new equation is exact.
$\Rightarrow$

$$f_x = \frac{1}{y}$$

$\Rightarrow$

$$f = \frac{x}{y} + g(y).$$

Hence

$$f_y = -\frac{x}{y^2} + g'(y) = -\frac{x}{y^2} + 1$$

$\Rightarrow g'(y) = 1 \Rightarrow g = y + c$
Therefore

$$f(x,y) = \frac{x}{y} + y + c$$

and the solution is $\frac{x}{y} + y = k$.

Example $\quad -ydx + xdy = 0$

$N = x \quad M = -y \ \Rightarrow \ M_y = -1$ and $N_x = 1$ Clearly this equation is not exact.
Multiply by u and get

$$-u(y)dx + u(x)dy = 0.$$

Then
$\mathrm{M}_y = u + yu_y$ and $\mathrm{N}_x = -u - u_x x$.

$$u + yu_y = -u - u_x x$$

Setting $u_y = 0$ yields $u = \frac{1}{x^2} \Rightarrow$

$$-\frac{y}{x^2}dx + \frac{1}{x}dy = 0$$

This new equation is exact.

$\Rightarrow$

$$f_x = -\frac{y}{x^2} \quad \text{and} \quad f_y = -\frac{y}{x^2}$$

$\Rightarrow$

$$f = \frac{y}{x} + h(x)$$

$$f_x = -\frac{y}{x^2} + h'(x) = -\frac{y}{x^2}$$

$\Rightarrow$

$$h' = 0$$

$\Rightarrow$

$$h = c \Rightarrow f = \frac{y}{x} + c$$

$\Rightarrow$ solution is

$$\frac{y}{x} = k$$

.