Ma 221

CHAPTER 4 - Linear Differential Equations

We shall now begin a detailed study of the second-order linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

Fundamental theory of second-order linear equations

The following theorem gives information concerning the existence of solutions of second-order linear differential equations. We shall accept it as valid without proof.

Theorem 1: Consider the Initial Value Problem

$$\text{D.E. } a(x)y'' + b(x)y' + c(x)y = f(x)$$

$$\text{I.C. } y(x_0) = y_0 \qquad y'(x_0) = y_0'$$

If $a(x),\ b(x),\ c(x),\ f(x)$ are all continuous functions in the interval I, where $x_0 \in I$ and $a(x) \neq 0$ for all x in I, then the IVP possesses a unique solution. This solution has a continuous derivative and is defined throughout I.

Example

$$D.E.\ a(x)y'' + b(x)y' + c(x)y = 0 \qquad \text{Homogeneous Equation}$$

$$\text{I.C. } y(x_0) = 0 \qquad y'(x_0) = 0$$

One solution is $y(x) \equiv 0$. Theorem 1 $\Rightarrow$ only solution is $y \equiv 0$.

We shall assume from now on that $a, b, c,$ and f are continuous in a common interval I and $a(x) \neq 0$ in I so that Theorem 1 holds.

Notation: Let

$$L[y] \equiv a(x)y'' + b(x)y' + c(x)y.$$

Then $L[2] = 2c(x)$

$L[3x] = 3b(x) + 3xc(x)$.

With this notation the second order differential equation $a(x)y'' + b(x)y' + c(y)y = f(x)$ can be written as $L[y]= f(x)$. The homogeneous case is when $f(x) = 0 \Rightarrow L[y] = 0$. This is called the homogeneous equation. If $f(x) \neq 0 \Rightarrow$ a nonhomogeneous equation.

$L[y]$ is called a linear operator because it has the following property.

Theorem 2:

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$$

where y_1 and y_2 are any twice differential functions and c_1 and c_2 are any constants.
Proof:

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= a(x)\ (c_1y_1 + c_2y_2)'' + b(x)\ (c_1y_1 + c_2y_2)' + c(x)\ (c_1y_1 + c_2y_2) \\ &= a(x)\ (c_1y_1'' + c_2y_2'') + b(x)\ (c_1y_1' + c_2y_2')\ + c(x)\ (c_1y_1 + c_2y_2) \\ &= c_1[a(x)y_1'' + b(x)y_1' + c(x)y_1] + c_2[a(x)y_2'' + b(x)y_2' + c(x)y_2] \end{aligned}$$

$\Rightarrow L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$

Properties of solutions of second order equations.

Theorem 3: If $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation $L[y] = 0$, then $y = c_1y_1(x) + c_2y_2(x)$ is also a solution.
Proof. $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$ from above. Since y_1 is a solution of $L[y] = 0 \Rightarrow L[y_1] = 0$. Similarly $L[y_2] = 0$.
Hence $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2] = 0 + 0 = 0 \Rightarrow y = c_1y_1 + c_2y_2$ is also a solution of $L(y) = 0$.

Example $y'' - 9y = 0$ e^{3x} and e^{-3x} are solutions. Theorem 3 $\Rightarrow$ $y = c_1e^{3x} + c_2e^{-3x}$ is also a solution.

Remark. We desire to be able to find the general solution of $L[y] = 0$. The above theorem tells us that if y_1 and y_2 are solutions, then $c_1y_1 + c_2y_2$ is a solution, but it does not tell us that this is the general solution. In order to know when one has a general solution it is necessary to introduce the concept of the linear independence of two functions.

Definition: Two functions $y_1(x)$ and $y_2(x)$ are called linearly dependent (LD) in an interval I if it is possible to find two constants c_1 and c_2, not both zero, so that

$$c_1y_1(x) + c_2y_2(x) = 0 \ \ \forall x \in I.$$

Two functions are called linearly independent (LI) if they are not linearly dependent, i.e., if

$$c_1y_1(x) + c_2y_2(x) = 0 \ \ \forall x \in I \Rightarrow c_1 = c_2 = 0.$$

Remark. If two functions are *LD* in *I* $\Rightarrow$ one of the functions is equal to a constant times the other in *I*.

Example (a) $x,\ 2x$ are *LD* in any interval *I*, since

$$(-2)x + (1)\ 2x = 0 \quad \forall x \in I$$

(b) $x^2,\ x$ are *LI* in any interval *I*, since

$$c_1x^2 + c_2x = 0 \quad \forall x \in I$$

is impossible because this equation has at most two real roots in *I*. Thus, we must have $c_1 = c_2 = 0$.

(c) Two functions are *LD* if one of them is the zero function. If $y_1 \equiv 0,$ then

$$c_1 y_1 + 0 \cdot y_2 = c_1 0 + 0 \cdot y \equiv 0 \ \forall x \in I$$

and any $c_1 \neq 0.$

(d) If $\lambda_1 \neq \lambda_2$, then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are LI for if

$$c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \equiv 0$$

$\Rightarrow$

$$c_1 \equiv -c_2 e^{(\lambda_2 - \lambda_1)x}.$$

But c_1 is a constant and therefore the last equation $\Rightarrow \lambda_1 = \lambda_2,$ which is a contradiction.

Facts from algebra needed in the proofs of the next theorems.

1. $\left.\begin{array}{l} d_1 x + d_2 y = d_3 \\ e_1 x + e_2 y = e_3 \end{array}\right\}$ has a unique solution $\Leftrightarrow \begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix} \neq 0$

   If $d_3 = e_3 = 0$ and $det \neq 0 \Rightarrow x = y = 0$ is the only solution.

2. $\left.\begin{array}{l} d_1 x + d_2 y = 0 \\ e_1 x + e_2 y = 0 \end{array}\right\}$ has nontrivial solution. $\Leftrightarrow \begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix} = 0$

Definition: The Wronskian of two differentiable functions y_1 and y_2 is defined to be

$$W[y_1(x),\ y_2(x)] \equiv \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \equiv y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Theorem 4. If $W[y_1(x), y_2(x)]$ is different from zero for at least one point in an interval I, then $y_1(x)$ and $y_2(x)$ are *LI* in I.

Proof. Suppose $y_1,\ y_2$ are *LD*. Then $\exists$ constants c_1, c_2, not both zero, such that

$$\left.\begin{array}{l} c_1 y_1(x) + c_2 y_2(x) \equiv 0 \\ c_1 y_1'(x) + c_2 y_2'(x) \equiv 0 \end{array}\right\}$$

By assumption these two equations have a nontrivial solution c_1, c_2 at each point x in I. Therefore the determinant of the coefficients (by 2) must be zero for each x. But the determinant of coefficients $= W[y_1(x), y_2(x)]$ and $W \neq 0$ for at least one point in I. $\Rightarrow y_1$ and y_2 are not *LD*.

Corollary. If y_1, y_2 are *LD* in $I \Rightarrow W[y_1(x),\ y_2(x)] \equiv 0$ in I.

Remark. Converse of Theorem 4 is not true in general, i.e., there exist functions which are *LI* in an

interval I and whose Wronskian is $\equiv 0$ in I.
However, if y_1 and y_2 are solutions of $L[y] = 0$ then the following converse holds.

Theorem 5. If $y_1(x),\ y_2(x)$ are LI solutions of $L[y] = 0$ in I, then $W[y_1(x),\ y_2(x)]$ is never zero in I.

Proof. If $W[y_1(x), y_2(x)] = 0$ for some $x_0 \in I$, then the equations

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0$$

have a nontrivial solution, i.e. $\exists\ c_1, c_2$ not both zero satisfying the system. For these values of c_1 and c_2 the function $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is a solution of $L[y] = 0$ and satisfies the initial conditions $y(x_0) = 0, y'(x_0) = 0$. However, by Theorem 1 the only solution of this problem is $y(x) \equiv 0 \Rightarrow$ $c_1 y_1(x) + c_2 y_2(x) \equiv 0\ \forall x \in I \Rightarrow y_1, y_2$ are LD. Contradiction! $\Rightarrow W[\ ]$ is never zero in I.

Corollary. The Wronskian of 2 solutions of $L[y] = 0$ is either identically zero (if solutions are LD) or never zero (if solutions are LI).

Theorem 6. If $y_1(x)$ and $y_2(x)$ are LI solutions of $L[y] = 0$, then $y = c_1 y_1 + c_2 y_2$ is the general solution of $L[y] = 0$.

Example e^{3x} and e^{-3x} are LI solutions of $y'' - 9y = 0 \Rightarrow$ general solution is $y = c_1 e^{3x} + c_2 e^{-3x}$.

Theorem 6 tells us that the problem of finding the general solution of $L[y] = 0$ is reduced to finding any two linearly independent solutions.

Example This example is a video slide show. Slide Example

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Question: Do two LI solutions of $L[y] = 0$ actually exist? The answer is given in the affirmative by the next theorem.

Theorem 7. $\exists$ two linear independent solutions of $L[y] = 0$.

Proof. Let $y_1(x)$ be the unique solution of $L[y] = 0$ with initial conditions $y_1(x_0) = 1, y_1'(x_0) = 0$, and $y_2(x)$ be the unique solution of $L[y] = 0$ with initial conditions $y_2(x_0) = 0, y_2'(x_0) = 1$. Note that y_1 and y_2 exist by Theorem 1. Now y_1 and y_2 are LI by Theorem 5 since

$$W[y_1(x_0), y_2(x_0)] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Theorem 8. If y_p is any particular solution of the nonhomogeneous equation $L[y] = f(x)$ and y_h is the general solution of the homogeneous equation $L[y] = 0$, then the general solution of $L[y] = f(x)$ is $y = y_p + y_h$.

Example Solve $y'' - 9y = e^x$
We know that $y_h = c_1e^{3x} + c_2e^{-3x}$. $y_p = ?$ Assume $y_p = Ae^x$

$\Rightarrow Ae^x - 9Ae^x = e^x \Rightarrow -8A = 1 \qquad A = -\frac{1}{8} \Rightarrow y_p = -\frac{1}{8}e^x$
$\Rightarrow y = c_1e^{3x} + c_2e^{-3x} - \frac{1}{8}e^x$ is the general solution.

Theorem 9. Principle of superposition. If y_1 is a solution of $L[y] = f_1$ and y_2 is a solution of $L[y] = f_2$, then $y = y_1 + y_2$ is a solution of $L[y] = f_1 + f_2$.

Example Solve $y'' - 9y = e^x + 5$. Before we found that $y = -\frac{1}{8}e^x$ was a particular solution of $y'' - 9y = e^x$. To find
a particular solution of $y'' - 9y = 5$ assume $y \equiv k \Rightarrow k = -\frac{5}{9}$. The general solution of equation is therefore

$$y = c_1e^{3x} + c_2e^{-3x} - \frac{1}{8}e^x - \frac{5}{9}.$$

Extension: If for $i = 1, 2, \ldots, n$, y_i is a solution of $L[y] = f_i$, then $\sum_{i=1}^n y_i$ is a solution of $L[y] = \sum_{i=1}^n f_i$.

Complex-valued Solutions

A complex-valued function f of a real variable x is a function of the form

$$f(x) = u(x) + iv(x)$$

where $u(x)$ and $v(x)$ are real functions and $i = \sqrt{-1}$.

Definition. If $f = u + iv$, u, v real functions, then f is continuous if u and v are continuous; f is differential if u and v are differential and

$$f'(x) = u'(x) + iv'(x).$$

Example a) $f(x) = 3x + ix^2 \Rightarrow f'(x) = 3 + 2ix$

b) $\frac{d}{dx}(3x + ix^2)^2 = 2(3x + ix^2)(3 + 2ix) = 2(9x - 2x^3 + 9ix^2)$

c) Let

$$E(x) = e^{ax}(\cos bx + i\sin bx)$$

Then

$$\begin{aligned} E'(x) &= ae^{ax}(\cos bx + i\sin bx) + e^{ax}(-b\sin bx + bi\cos bx) \\ &= e^{ax}[a(\cos bx + i\sin bx) + bi(\cos bx + i\sin bx)] \\ &= e^{ax}[a + bi](\cos bx + i\sin bx). \end{aligned}$$

Hence

$$E'(x) = (a+bi)E(x).$$

Based on this we define the complex exponential via

$$e^{(a+bi)x} = e^{ax}\cos bx + ie^{ax}\sin bx$$

$a = 0 \Rightarrow$

$$e^{bix} = \cos bx + i\sin bx.$$

This is called Euler's formula. Hence

$$e^{(a+bi)x} = e^{ax} \bullet e^{bix}.$$

Example $y = e^{ix}$ satisfies $y'' + y = 0$ since $y' = ie^{ix}$ $y'' = -e^{ix} \Rightarrow -e^{ix} + e^{ix} = 0.$

The theorem below gives the connection between real and complex solutions of a linear differential equation with real coefficients.

Theorem 1. Consider the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

where $a(x)$, $b(x)$, and $c(x)$ are real functions. The complex function $y = u + iv$, where u and v are real, is a solution of this equation $\Leftrightarrow$ u and v are solutions.

Proof. As before we denote the equation by $L[y] = 0$. It is easily shown that $L[y] = L[u] + iL[v]$ where $L[u]$ and $L[v]$ are real. Therefore y is a solution $\Leftrightarrow$ $L[y] = L[u] + iL[v] \equiv 0$. Since a complex number is zero $\Leftrightarrow$ its real and imaginary parts are zero,
$\Rightarrow L[y] = 0 \Leftrightarrow L[u] = 0$ and $L[v] = 0 \Leftrightarrow u$ and v solutions.

Example $y = e^{ix}$ is a solution of $y'' + y = 0$. Since $e^{ix} = \cos x + i\sin x \Rightarrow \cos x$ and $\sin x$ are solutions. This is easily verified.

Homogeneous Linear Equations with Constant Coefficients

We shall now discuss the problem of solving the homogeneous equation

$$ay'' + by' + cy = 0 \ (*)$$

where a, b and c are real constants and $a \neq 0$.

Possible candidates for a solution are x and powers of x. These are no good. $\ln x$ is also no good. We shall try $e^{\lambda x}$. If $y = e^{\lambda x}$ is a solution of (*)
$\Rightarrow a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$. This is to be a solution $\forall x$. $\Rightarrow$

$$a\lambda^2 + b\lambda + c = 0.$$

This equation for λ is called the *auxiliary* or *characteristic* equation.

It has the solution $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ $\qquad \Delta = b^2 - 4\,ac$

There are three possibilities:

(1) $\Delta > 0$ two real, distinct roots

(2) $\Delta = 0$ one real root, repeated
(3) $\Delta < 0$ two imaginary roots which are the complex conjugates of each other, i.e. if $\lambda_1 = \alpha + i\beta \Rightarrow$ $\lambda_2 = \alpha - i\beta$
We shall now discuss the three cases in detail.

Case 1. $\Delta > 0$. There are two real distinct roots λ_1, λ_2, where $\lambda_1 \neq \lambda_2$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$\Rightarrow e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are both solutions of the differential equation. These functions are LI, $\Rightarrow$ general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

where λ_1 and λ_2 are both real and $\lambda_1 \neq \lambda_2$.

Example $2y'' - y' - 3y = 0$

$\Rightarrow 2\lambda^2 - \lambda - 3 = 0$
or
$(2\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda_1 = -1 \;\; \lambda_2 = \frac{3}{2} \;\; \Rightarrow y = c_1 e^{-x} + c_2 e^{+\frac{3}{2}x}$.

Case 2. $\Delta = 0$. There is one real, repeated root $\lambda_1 = -\dfrac{b}{2a} \Rightarrow e^{\lambda_1 x}$ is a solution. We need a second LI solution. To find it we shall use the method of variation of parameters. We seek a solution of the form

$$y = v(x)e^{\lambda_1 x},$$

where $v(x)$ is a function to be determined. Now

$$y' = v' e^{\lambda_1 x} + v\lambda_1 e^{\lambda_1 x}$$

and

$$y'' = v'' e^{\lambda_1 x} + 2v'\lambda_1 e^{\lambda_1 x} + v\lambda_1^2 e^{\lambda_1 x}$$

$\Rightarrow$

$$av''(x)e^{\lambda_1 x} + 2av'\lambda_1 e^{\lambda_1 x} + av\lambda_1^2 e^{\lambda_1 x} + bv' e^{\lambda_1 x} + bv\lambda_1 e^{\lambda_1 x} + cve^{\lambda_1 x} = 0$$

$\Rightarrow$

$$av'' + \underbrace{(2a\lambda_1 + b)}v' + \underbrace{(a\lambda_1^2 + b\lambda_1 + c)}v = 0$$

$\Rightarrow v'' = 0$ (why?) $\Rightarrow$

$$v = c_1 + c_2 x$$

$\Rightarrow$

$$y = ve^{\lambda_1 x} = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

is a solution of the differential equation in the case where there exists one repeated root λ_1. Since $e^{\lambda_1 x}$

and $xc^{\lambda_1 x}$ are LI $\Rightarrow$ this is the general solution.

Example $y'' - 4y' + 4y = 0$
$\lambda^2 - 4\lambda + 4 = 0$ or $(\lambda - 2)^2 = 0 \Rightarrow$ one real, repeated root $\lambda = 2. \Rightarrow$

$$y = c_1 e^{2x} + c_2\ xe^{2x}.$$

Case 3. $\Delta < 0$ 2 complex roots

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{i\sqrt{4ac - b^2}}{2a}$$

$\Rightarrow$

$$\lambda_1 = \alpha + i\beta = -\frac{b}{2a} + i\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}$$
$$\lambda_2 = \alpha - i\beta = -\frac{b}{2a} - i\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}$$

where α and β are real numbers.
$\Rightarrow$ two complex solutions. $e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$ and $e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$

Since the differential equation has real coefficients, $\Rightarrow$ real and imaginary parts of above are solutions, i.e.,
$e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are both solutions in this case. These are LI functions.
$\Rightarrow$ the solution is

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x) \quad \text{where } A, B \text{ real constants.}$$

Example

$$32y'' - 40y' + 17y = 0$$

$$32\lambda^2 - 40\lambda + 17 = 0$$

$$\lambda = \frac{40 \pm \sqrt{1600 - 4(32)(17)}}{2(32)} = \frac{40 \pm \sqrt{1600 - 2176}}{2(32)} = \frac{40 \pm i\sqrt{576}}{2(32)} = \frac{5 \pm \sqrt{9}}{8} = \frac{5}{8} \pm \frac{3}{8}\ i$$

Thus

$$\lambda_1 = \frac{5}{8} + \frac{3}{8}i \text{ and } \lambda_2 = \frac{5}{8} - \frac{3}{8}i$$

$\Rightarrow$

$$y = e^{\frac{5}{8}x}(A\cos\frac{3}{8}x + B\sin\frac{3}{8}x)$$

Example Write down a second order homogeneous linear differential equation with real constant coefficients whose solutions are

$$\frac{1}{2}e^{-2x}\cos 3x \text{ and } \frac{3e^{-2x}}{4}\sin 3x.$$

$\Rightarrow \alpha = -2 \quad \beta = 3$ so that $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$.

$\Rightarrow$

$$\begin{aligned} p(\lambda) &= [\lambda - (-2+3i)][\lambda - (-2-3i)] \\ &= [\lambda + 2 - 3i][\lambda + 2 + 3i] \\ &= \lambda^2 + (2+3i)\lambda + (2-3i)\lambda + 4 + 9 \\ &= \lambda^2 + 4\lambda + 13 \end{aligned}$$

(Check: $\lambda = \dfrac{-4 \pm \sqrt{16 - 4(1)(13)}}{2} = \dfrac{-4 \pm bi}{2} = -2 \pm 3i$)

$\Rightarrow$ equation is

$$y'' + 4y' + 13y = 0$$

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Undetermined Coefficients

Let us now consider the problem of solving

$$ay'' + by' + cy = f(x) \quad a \neq 0 \quad (*)$$

a, b, c real constants. We know that the general solution is $y = y_h + y_p$, where

$$y_h = \text{the solution of the homogeneous equation}$$

and

$$y_p = \text{a particular solution of the equation}$$

We know how to find y_h . We shall now discuss ways of finding y_p for certain special functions $f(x)$.

1. $f(x) = Ke^{\alpha x}$ K **constant**, α **constant**.

Thus we seek y_p for

$$ay'' + by' + cy = Ke^{\alpha x}.$$

Due to the exponential form of $f(x)$ we seek y_p of the form

$$y_p = Ae^{\alpha x}$$

$A = ?$ The differential equation (*) $\Rightarrow$

$$(a\alpha^2 + b\alpha + c)Ae^{\alpha x} = Ke^{\alpha x}$$

$\Rightarrow$

$$A = \frac{K}{a\alpha^2 + b\alpha + c}$$

$\Rightarrow$

$$y_p = \frac{Ke^{\alpha x}}{a\alpha^2 + b\alpha + c}$$

The above is a particular solution provided the denominator is non-zero. Note that the denominator is $p(\lambda) = a\lambda^2 + b\lambda + c$ with $\lambda = \alpha$. This is the characteristic polynomial with $\lambda = \alpha$.

If $p(\alpha) = 0$, $\Rightarrow$ we do not have a y_p. However, $p(\alpha) = 0 \Rightarrow \alpha$ is a root of characteristic equation. $\Rightarrow e^{\alpha x}$ is solution of the homogeneous equation, and therefore $Ae^{\alpha x}$ cannot be a solution of the nonhomogeneous equation. If $p(\alpha) = 0$, we try

$$y_p = Axe^{\alpha x}$$

$\Rightarrow y_p' = A\alpha xe^{\alpha x} + Ae^{\alpha x}$ and $y_p'' = A\alpha^2 xe^{\alpha x} + A\alpha e^{\alpha x} + A\alpha e^{\alpha x} = A\alpha^2 xe^{\alpha x} + 2A\alpha e^{\alpha x}$

Substitution into the differential equation $(*) \Rightarrow$

$$Axe^{\alpha x}[a\alpha^2 + b\alpha + c] + Ae^{\alpha x}[2a\alpha + b] = Ke^{\alpha x}$$

$\Rightarrow$

$$A = \frac{K}{2a\alpha + b}$$

$\Rightarrow$

$$y_p = \frac{Kxe^{\alpha x}}{2a\alpha + b} \quad \text{if} \quad p(\alpha) = 0$$

provided, of course, that $2a\alpha + b \neq 0$. Note that $p(\lambda) = a\lambda^2 + b\lambda + c \Rightarrow p'(\lambda) = 2a\lambda + b$

$\Rightarrow$

$$y_p = \frac{Kxe^{\alpha x}}{p'(\alpha)} \text{ when } p(\alpha) = 0 \text{ and } p'(\alpha) \neq 0$$

If $p(\alpha) = 0$ and $p'(\alpha) = 0 \Rightarrow$ above y_p is no good. But $p'(\alpha) = 0 \Rightarrow 2a\alpha + b = 0 \Rightarrow \alpha = -\frac{b}{2a}$. $\Rightarrow \alpha$ is a double (repeated) root of $a\lambda^2 + b\lambda + c = 0$. Hence both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the homogeneous equation if $p(\alpha) = p'(\alpha) = 0$, and these cannot therefore be solutions of the nonhomogeneous equation. If $p(\alpha) = p'(\alpha) = 0$ we try

$$y_p = Ax^2e^{\alpha x}.$$

Differentiating and substituting into the equation leads to

$\Rightarrow$

$$A = \frac{K}{2a} = \frac{K}{p''(\alpha)}$$

since $p'(\lambda) = 2a\lambda + b \quad \Rightarrow \quad p''(\lambda) = 2a$

Thus

$$y_p = \frac{K}{p''(\alpha)}x^2e^{\alpha x} \quad \text{if} \quad p(\alpha) = p'(\alpha) = 0.$$

$p''(\alpha) \neq 0$ since $a \neq 0$ by assumption.

Summary: A particular solution of $L[y] = ke^{\alpha x}$ is

$$y_p = \frac{Ke^{\alpha x}}{p(\alpha)} \quad \text{if } p(\alpha) \neq 0$$
$$y_p = \frac{Kxe^{\alpha x}}{p'(\alpha)} \quad \text{if } \ p(\alpha) = 0, p'(\alpha) \neq 0$$
$$y_p = \frac{K}{p''(\alpha)}x^2e^{\alpha x} \quad \text{if } \ p(\alpha) = p'(\alpha) = 0$$

Example (a) $y'' - 5y' + 4y = 2e^x$
Homogeneous solution: $p(\lambda) = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1) \quad \Rightarrow \lambda = 4,\ 1 \Rightarrow y_h = c_1e^x + c_2e^{4x}$
Now to find a particular solution for $2e^x$. $\Rightarrow \alpha = 1 \quad p(1) = 0 \quad$ Since $p'(\lambda) = 2\lambda - 5$
$p'(1) = 2 - 5 = -3 \neq 0$
$\Rightarrow$

$$y_p = \frac{kxe^{\alpha x}}{p'(\alpha)} = \frac{2xe^x}{-3}$$

$\Rightarrow$

$$y = y_h + y_p = c_1e^x + c_2e^{4x} - \frac{2}{3}xe^x$$

(b) $y'' - 5y' + 4y = 3 + 2e^x$
$y_h = c_1e^x + c_2e^{4x} \quad p(\lambda) = \lambda^2 - 5\lambda + 4$
Consider $y'' - 5y' + 4y = 3$
$3 = ke^{\alpha x}$ with $k = 3 \ \ \alpha = 0 \ \ p(0) = 4 \neq 0 \quad \Rightarrow y_p' = \frac{3}{4}$
$\Rightarrow$

$$y(x) = c_1e^x + c_2e^{4x} + \frac{3}{4} - \frac{2}{3}xe^x$$

(c) Let us now consider an IVP for this DE.

$$y'' - 5y' + 4y = 3 + 2e^x \ \ y(0) = 1 \ \ y'(0) = 0$$

$$y(0) = c_1 + c_2 + \frac{3}{4} = 1$$
$$y'(x) = c_1e^x + 4c_2e^{4x} - \frac{2}{3}xe^x - \frac{2}{3}e^x$$
$$y'(0) = c_1 + 4c_2 - \frac{2}{3} = 0$$

Thus we have the two equations below for c_1 and c_2 :

$$c_1 + c_2 = \frac{1}{4}$$
$$c_1 + 4c_2 = \frac{2}{3}$$

, Solution is: $[c_1 = \frac{1}{9}, c_2 = \frac{5}{36}]$ so

$$y(x) = \frac{1}{9}e^x + \frac{5}{36}e^{4x} + \frac{3}{4} - \frac{2}{3}xe^x$$

SNB Check:

$y'' - 5y' + 4y = 3 + 2e^x$

$y(0) = 1$, , Exact solution is: $\{\frac{1}{9}e^x + \frac{5}{36}e^{4x} - \frac{2}{3}xe^x + \frac{3}{4}\}$

$y'(0) = 0$

2. $f(x) = k$ **cos** βx **or** $f(x) = k$ **sin** βx

For example, we seek a particular solution of

$$L[y] = ay'' + by' + cy = k\cos\beta x$$

We shall use the complex exponential to solve for y_p . Recall

$$ke^{i\beta x} = k\cos\beta x + ik\sin\beta x.$$

Hence we consider also the equation

$$L[v] = av'' + bv' + cv = k\sin\beta x$$

By multiplying this last equation by i and adding the result to $L[y] \Rightarrow$

$$L[y] + iL[v] = k\cos\beta x + ik\sin\beta x = ke^{i\beta x}.$$

But $iL[v] = L[iv]$, since$L$ is linear. Hence if we let $w = y + iv \Rightarrow$ the equation

$$aw'' + bw' + cw = ke^{i\beta x}$$

or

$$L[y] + iL[v] = L[y] + L[iv] = L[y + iv] = L[w] = ke^{i\beta x}$$

and therefore we have the complex equation $L[w] = ke^{i\beta x}$ for w. To find w_p for this we use the formulas derived above. Then we find y_p from $y_p = \text{Re } w_p =$ real part of w_p. For $f(x) = k\sin\beta x$ we have $y_p = \text{Im } w_p$ =imaginary part of w_p.

Example Find a particular solution of

$$y'' + 7y' + 12y = 3\cos 2x$$

Let $w = y + iv \Rightarrow$ find w_p for $w'' + 7w' + 12w = 3e^{2ix}$. Now $p(\lambda) = \lambda^2 + 7\lambda + 12 \Rightarrow$
$p(\alpha) = p(2i) = (2i)^2 + 7(2i) + 12 = -4 + 14i + 12 \neq 0$
$\Rightarrow$

$$w_p = \frac{3e^{2ix}}{p(2i)} = \frac{3e^{2ix}}{8 + 14i}.$$

$y_p = \text{Re } w_p = ?$

To find y_p we shall rationalize the denominator.

$$
\begin{aligned}
w_p &= \frac{3e^{2ix}}{8+14i} \times \frac{8-14i}{8-14i} \\
&= \frac{3(8-14i)e^{2ix}}{64+196} \\
&= \frac{3(8-14i)e^{2ix}}{260} \\
&= \frac{3}{260}(8-14i)[\cos 2x + i\sin 2x] \\
&= \frac{3}{260}[8\cos 2x + 14\sin 2x] + \frac{3}{260}i[8\sin 2x - 14\cos 2x]
\end{aligned}
$$

Thus

$$y_p = \operatorname{Re} w_p = \frac{3}{260}[8\cos 2x + 14\sin 2x]$$

Example

$$y'' + 4y = 3\sin 2x$$

$\Rightarrow$

$$w'' + 4w = 3e^{2ix}$$

$p(\lambda) = \lambda^2 + 4 \;\; \Rightarrow p(2i) = 0$ and $p'(\lambda) = 2\lambda$. Now $p'(2i) \neq 0$

$\Rightarrow$

$$
\begin{aligned}
w_p &= \frac{3xe^{2ix}}{p'(2i)} = \frac{3xe^{2ix}}{4i} \times \frac{i}{i} = -\frac{3}{4}ixe^{2ix} = -\frac{3}{4}ix[\cos 2x + i\sin 2x] \\
&= -\frac{3}{4}xi\cos 2x + \frac{3}{4}x\sin 2x
\end{aligned}
$$

$\Rightarrow$

$$y_p = \operatorname{Im} w_p = -\frac{3}{4}x\cos 2x$$

Example $y'' + 7y' + 12y = 3\cos 2x$ again.

Let $y_p = A\cos 2x + B\sin 2x$

$y_p' = -2A\sin 2x + 2B\cos 2x \qquad y_p'' = 4A\cos 2x - 4B\sin 2x$

$\Rightarrow$

$$-4A\cos 2x - 4B\sin 2x - 14A\sin 2x + 14B\cos 2x + 12A\cos 2x + 12B\sin 2x = 3\cos 2x$$

$\Rightarrow$

$$\cos 2x[8A + 14B] + \sin 2x[8B - 14A] = 3\cos 2x$$

$\Rightarrow$

$$8A + 14B = 3 \;\; 8B - 14A = 0 \;\; \Rightarrow \;\; B = \frac{7}{4}A$$

$8A + \frac{7}{2}(7)A = 3 \quad 8A + \frac{49}{2}A = 3 \;\Rightarrow\; \frac{16+49}{2}A = 3 \quad A = \frac{6}{65} \;\; \Rightarrow B = \frac{21}{130}$

$\Rightarrow\; y_p = \frac{6}{65}\cos 2x + \frac{21}{130}\sin 2x$ as before.

III. $f(x) = B_0 + B_1x + \cdots + B_nx^n$ **polynomial**.

We want y_p for

$$ay'' + by' + cy = B_0 + B_1 x + \cdots + B_n x^n$$

We try a solution of the form

$$y_p = Q_n(x) = A_0 + A_1 x + \cdots + A_n x^n$$

If $p(0) \neq 0,$ then when we substitute Q_n into the equation we will get a polynomial of degree n and we can determine A_k' s by equating coefficients of like powers of x. If $p(0) = 0,$ but $p'(0) \neq 0$ use $y_p = xQ_n(x)$. Similarly if $p(0) = p'(0) = 0$ take $y_p = x^2 Q_n(x)$.

Example

$$y'' + 3y' = 2x^2 + 3x$$

In this example the right hand side is a polynomial of degree 2.

$p(\lambda) = \lambda^2 + 3\lambda$ so $p(0) = 0.$ $p'(\lambda) = 2\lambda + 3$ and $p'(0) \neq 0$

$\Rightarrow$

$$y_p = xQ_2(x) = x(A_0 + A_1 x + A_2 x^2) = A_0 x + A_1 x^2 + A_2 x^3$$

$\Rightarrow$

$$y_p' = A_0 + 2A_1 x + 3A_2 x^3 \Rightarrow y_p'' = 2A_1 + 6A_2 x$$

The differential equation $\Rightarrow$

$$2A_1 + 6A_2 x + 3A_0 + 6A_1 x + 9A_2 x^2 = 2x^2 + 3x$$

$\Rightarrow$

$$2A_1 + 3A_0 = 0 \text{ and } 6A_2 + 6A_1 = 3 \text{and } 9A_2 = 2.$$

$\Rightarrow A_2 = \frac{2}{9}$ $A_2 + A_1 = \frac{1}{2}$ $\frac{2}{9} + A_1 = \frac{1}{2} \Rightarrow A_1 = \frac{1}{2} - \frac{2}{9} = \frac{9-4}{18} = \frac{5}{18}$

$2\left(\frac{5}{18}\right) + 3A_0 = 0$ $A_0 = -\frac{10}{18(3)} = -\frac{5}{27} \Rightarrow$

$$y_p = -\frac{5}{27}x + \frac{5}{18}x^2 + \frac{2}{9}x^3$$

IV. $f(x) = (B_0 + B_1 x + \cdots + B_n x^n)e^{\alpha x}$

We want a particular solution for the DE

$$ay'' + by' + cy = (B_0 + B_1 x + \cdots + B_n x^n)e^{\alpha x}$$

We seek a solution of the form

$$y_p = Q_n(x)e^{\alpha x} \text{ if } p(\alpha) \neq 0$$
$$y_p = xQ_n(x)^{\alpha x} \text{ if } p(\alpha) = 0, \text{ and } p'(\alpha) \neq 0$$
$$y_p = x^2 Q_n(x)e^{\alpha x} \text{ if } p(\alpha) = p'(\alpha) = 0$$

Additional Examples

Example Solve

$$y'' + y = x\cos x - \cos x$$

Solution: Note that $y_h = C_1 \cos x + C_2 \sin x$.

First we will find a particular solution for $\cos x$. Consider

$$y'' + y = -\cos x$$

and

$$v'' + v = -\sin x$$

Multiply the second equation by i and add it to the first equation.
Letting $w = y + iv,$ we get

$$w'' + w = -(\cos x + i\sin x) = -e^{ix}$$

Since $p(\lambda) = \lambda^2 + 1$ and $p(i) = 0, p'(\lambda) = 2\lambda,$ so $p'(i) = 2i \neq 0$

$$w_{p_1} = -\frac{xe^{ix}}{2i} = \frac{1}{2}ixe^{ix}$$

Hence

$$y_{p_1} = \operatorname{Re} w_{p_1} = -\frac{x}{2}\sin x$$

Now we shall find a particular solution for $x\cos x.$ Consider

$$y'' + y = x\cos x$$

and

$$v'' + v = x\sin x$$

Multiplying the second equation by $i,$ adding it to the first equation and letting $w = y + iv,$ we have

$$w'' + w = x(\cos x + i\sin x) = xe^{ix}$$

Since e^{ix} is a homogeneous solution and xe^{ix} corresponds to a right hand side of $e^{ix},$ we let

$$w_{p_2} = (A_1 x + A_2 x^2)e^{ix}$$

to deal with a right side of the form xe^{ix}.

$$w'_{p_2} = (A_1 + 2A_2 x)e^{ix} + i(A_1 x + A_2 x^2)e^{ix}$$
$$w''_{p_2} = 2A_2 e^{ix} + 2i(A_1 + 2A_2 x)e^{ix} - (A_1 x + A_2 x^2)e^{ix}$$

Substituting into the DE leads to

$$2A_2 e^{ix} + 2i(A_1 + 2A_2 x)e^{ix} = xe^{ix}$$

Therefore

$$2A_2 + 2iA_1 = 0$$
$$4iA_2 = 1 \text{ or } A_2 = \frac{1}{4i} = -\frac{i}{4}$$

Then

$$A_1 = -\frac{1}{i}A_2 = \frac{1}{4}$$

$$w_{p_2} = \frac{1}{4}xe^{ix} - \frac{i}{4}x^2 e^{ix} = \left(\frac{1}{4}x - \frac{i}{4}x^2\right)(\cos x + i\sin x)$$

$$y_{p_2} = \operatorname{Re} w_{p_2} = \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x$$

Thus

$$y = y_h + y_{p_1} + y_{p_2} = C_1\cos x + C_2\sin x - \frac{x}{2}\sin x + \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x$$

Example Consider the equation

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x \qquad (*)$$

(a) Find the general solution to this equation.

$y'' - 3y' + 2y = 0$ characteristic equation: $r^2 - 3r + 2 = 0;\quad r = 1, 2$

$y_h = C_1e^x + C_2e^{2x}$

$y_p = Ae^{-x} + B\sin 3x + C\cos 3x$ [note: e^{-x} is not a homogeneous solution.]
$y_p' = -Ae^{-x} + 3B\cos 3x - 3C\sin 3x$
$y_p'' = Ae^{-x} - 9B\sin 3x - 9C\cos 3x$

After plugging y_p into the given DE, $y_p'' - 3y_p' + 2y_p$ and equating the coefficients:

Solution is: $y(x) = \frac{1}{2}e^{-x} + \frac{9}{13}\sin 3x + \frac{7}{13}\cos 3x + C_1e^x + C_2e^{2x}$

(b) Find the solution to (*) which also satisfies the initial conditions

$$y(0) = 1,\ y'(0) = 2.$$

$$y(x) = \frac{1}{2}e^{-x} + \frac{9}{13}\sin 3x + \frac{7}{13}\cos 3x + C_1e^x + C_2e^{2x}$$

$y(0) = \frac{27}{26} + C_1 + C_2 = 1 \qquad C_1 = -\frac{1}{26} - C_2$

$y'(x) = -\frac{1}{2}e^{-x} + \frac{27}{13}\cos 3x - \frac{21}{13}\sin 3x + C_1e^x + 2C_2e^{2x}$

$y'(0) = \frac{41}{26} + C_1 + 2C_2 = 2 \qquad \frac{41}{26} - \frac{1}{26} - C_2 + 2C_2 = 2$

$\frac{40}{26} + C_2 = 2 \qquad C_2 = 2 - \frac{40}{26} = \frac{6}{13}$

$\frac{27}{26} + C_1 + \frac{6}{13} = 1 \qquad C_1 = 1 - \frac{39}{26} = -\frac{1}{2}$

$C_2 = \frac{6}{13}$ and $C_1 = -\frac{1}{2}$

Solution is:

$$y(x) = \frac{1}{2}e^{-x} + \frac{9}{13}\sin 3x + \frac{7}{13}\cos 3x - \frac{1}{2}e^x + \frac{6}{13}e^{2x}$$

Example These examples are video slide shows. Slide Example 1 Slide Example 2

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Variation of Parameters

Let us now consider the non-homogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

where a, b, c, f are continuous *functions* in some internal I and $a(x) \neq 0\ \forall x \in I$. Note we are not assuming that a, b, and c are constants. We seek y_p, a particular solution. We shall use the method of variation of parameters.

If $y_1(x)$ and $y_2(x)$ are two (known) LI solutions of the homogeneous equation $\Rightarrow$

$$y_h = c_1 \; y_1(x) + c_2 \; y_2(x).$$

To find y_p we shall replace c_1 and c_2 by unknown functions of x and seek to determine these functions. Hence let

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$$

Substitution of the above into the differential equation $\Rightarrow$ only one condition for v_1 and v_2. We may therefore impose another condition arbitrarily but in such a manner as to simplify things.
Now

$$y_p' = v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2$$

If we require

$$v_1' \; y_1 + v_2' \; y_2 \equiv 0 \quad (*)$$

then no second derivatives of v_1 and v_2 will appear in y_p''. We therefore make this one condition. The other comes from the differential equation. Now $(*) \Rightarrow$

$$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''$$

Substituting into the differential equation implies

$$v_1(\underbrace{ay_1'' + by_1' + cy_1}) + v_2(\underbrace{ay_2'' + by_2' + cy_2}) + av_1'y_1' + av_2'y_2' = f(x)$$

Since the "lower bracketed" quantities are zero, this last equation $\Rightarrow$

$$v_1'y_1' + v_2'y_2' = \frac{f(x)}{a(x)}$$

.
This is a second condition for v_1' and v_2'.
$\Rightarrow$ we have found two equations to determine v_1, v_2 namely

$$v_1'y_1 + v_2'y_2 \equiv 0$$

and

$$v_1'y_1' + v_2'y_2' \equiv \frac{f(x)}{a(x)}$$

These two equations can be solved for v_1', v_2' provided

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

However, the above is the Wronskian of y_1 , y_2 and is never zero since y_1 and y_2 are LI.
$\Rightarrow$

$$v_1 = -\int \frac{y_2 f(x)}{a(x) \; W[y_1, y_2]} dx$$

and

$$v_2 = \int \frac{y_1 f(x)}{a(x) \; W[y_1, y_2]} dx$$

Note: It is not necessary to remember these formulas. One can usually solve the two equations for v_1' and v_2' directly and with ease. See the examples below.

The particular solution to non-homogeneous equation is

$$y_p = v_1(x)\ y_1(x) + v_2(x)\ y_2(x)$$

with v_1 and v_2 given by the above expressions.

Example

$$y'' + y = \sec x$$

⇒ let $y_1 = \cos x$ and $y_2 = \sin x,$ since these are the two LI homogeneous solutions. Then we take

$$y_p = v_1(x)\cos x + v_2(x)\sin x$$

The two conditions given above ⇒

$$v_1'\cos x + v_2'\sin x = 0$$
$$-v_1'\sin x + v_2'\ \cos x = \sec x.$$

Note that $W[y_1, y_2] = \cos^2 x + \sin^2 x = 1.$

Solving the first equation for v_1' we have

$$v_1' = -\frac{v_2'\sin x}{\cos x}$$

The second equation the implies

$$\frac{v_2'\sin^2 x}{\cos x} + v_2'\ \cos x = \sec x$$

Or

$$v_2'[\sin^2 x + \cos^2 x] = 1$$

Hence $v_2' = 1$ and

$$v_2 = \int 1 dx = x$$

From the first equation we

$$v_1 = -\int \frac{\sin x}{\cos x} dx = \ln|\cos x|$$

There is no need to include constants of integration, since these just lead to homogeneous solutions in y_p.

⇒

$$y_p = \ln|\cos x|\cos x + x\sin x.$$

Hence we get finally that

$$y = y_h + y_p = c_1 \cos x + c_2 \sin x + \ln|\cos x| \cos x + x \sin x$$

Example This example is a video slide show. Slide Example

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Example

Given that x and x^4 are homogeneous solutions of the equation

$$x^2 y'' - 4xy' + 4y = x^4 + x^2 \qquad x > 0$$

find the general solution of this equation.

Solution

Since we have that $\{x^4, x\}$ is a fundamental solution set, we seek a particular solution of the form

$$y_p(x) = v_1(x)x + v_2(x)x^4$$

The equations for v_1' and v_2' are

$$v_1' x + v_2' x^4 = 0$$

$$v_1' + 4v_2' x^3 = \frac{f}{a} = x^2 + 1$$

Note: The Wronskian is $\begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix} = 4x^4 - x^4 = 3x^4 \neq 0$ for

The first equation yields

$$v_1' = -v_2' x^3$$

The second equation then yields

$$v_2' = \frac{1}{3}\frac{x^2+1}{x^3} = \frac{1}{3}\frac{1}{x} + \frac{1}{3}\frac{1}{x^3}$$

Thus

$$v_2(x) = \int \left(\frac{1}{3}\frac{1}{x} + \frac{1}{3}\frac{1}{x^3} \right) dx = \frac{1}{3}\ln x - \frac{1}{6x^2}$$

Then

$$v_1' = -\left(\frac{1}{3}\frac{1}{x} + \frac{1}{3}\frac{1}{x^3} \right) x^3 = -\frac{1}{3}(x^2 + 1)$$

And we have

$$v_1(x) = -\frac{1}{3}\int (x^2 + 1) dx = -\frac{1}{9}x^3 - \frac{1}{3}x$$

Thus

$$\begin{aligned} y_p(x) &= \left(\frac{1}{3}\ln x - \frac{1}{6x^2} \right) x^4 + \left(-\frac{1}{9}x^3 - \frac{1}{3}x \right) x \\ &= \frac{1}{18}x^2(-2x^2 - 9 + 6(\ln x)x^2) \end{aligned}$$

The general solution is, then,

$$y(x) = \frac{1}{18}x^2(-2x^2 - 9 + 6(\ln x)x^2) + C_1x + C_2x^4$$

SNB check $x^2y'' - 4xy' + 4y = x^4 + x^2$, Exact solution is: $C_1x - \frac{1}{2}x^2 - \frac{1}{9}x^4 + C_2x^4 + \frac{1}{3}x^4\ln x$

Example Solve the equation

$$y'' - 2y' + y = 3e^t + \frac{e^t}{1+t^2}$$

Solution: $p(r) = r^2 - 2r + 1 = (r-1)^2$. Thus $r = 1$ is a repeated root and

$$y_h = c_1e^t + c_2te^t$$

Since $p'(r) = 2r - 2,$ then $p(1) = p'(1) = 0$ and a particular solution for $3e^t$ is

$$y_{p_1} = \frac{Kt^2e^{\alpha t}}{p''(\alpha)} = \frac{3t^2e^t}{2}$$

To find a particular solution for $\frac{e^t}{1+t^2}$ we use the Method of Variation of Parameters.

$$y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$$

Here $y_1 = e^t$ and $y_2 = te^t$ so

$$y_p = v_1(t)e^t + v_2(t)\ te^t$$

The equations for v_1' and v_2' are

$$v_1'e^t + v_2'te^t = 0$$

$$v_1'e^t + v_2'(e^t + te^t) = \frac{f}{a} = \frac{e^t}{1+t^2}$$

since $a = 1.$ We many cancel the e^t in both equations.

$$v_1' + v_2't = 0$$

$$v_1' + v_2'(1+t) = \frac{1}{1+t^2}$$

Then

$$v_1' = -v_2't$$

Using this in the second equation we have

$$v_2' = \frac{1}{1+t^2}$$

and therefore

$$v_2 = \tan^{-1}t$$

Then

$$v_1' = -\frac{t}{1+t^2}$$

so

$$v_2 = -\frac{1}{2}\ln(1+t^2)$$

Then

$$y_{p_2} = v_1(x)e^t + v_2(x)te^t = -\frac{1}{2}e^t\ln(1+t^2) + te^t\tan^{-1}t$$

Finally

$$\begin{aligned} y &= y_h + y_{p_1} + y_{p_2} \\ &= c_1e^t + c_2te^t + \frac{3t^2e^t}{2} - \frac{1}{2}e^t\ln(1+t^2) + te^t\tan^{-1}t \end{aligned}$$

Euler's Equation

The differential equation

$$x^2y'' + pxy' + qy = f(x) \quad (1)$$

where p and q are constants is called Euler's Equation (or the Cauchy-Euler Equation).

Consider the homogeneous case

$$x^2y'' + pxy' + qy = 0. \quad (2)$$

Once we find y_h then we can find y_p for (1) by variation of parameters. We consider (2) only for the case $x > 0$ so that coefficient on y'' does not vanish. Notice that each term contains some power of x if we try $y = x^m$. Hence we seek a homogeneous solution of the form

$$y_h = x^m$$

and shall try to determine m so that x^m is a solution. Now

$$y_h' = mx^{m-1} \text{ and } y_h'' = m(m-1)x^{m-2}$$

so that the differential equation ⇒

$$x^m[m(m-1) + pm + q] = 0 \Rightarrow x^m[m^2 + m(p-1) + q] = 0.$$

Since $x^m \neq 0 \Rightarrow$

$$m^2 + m(p-1) + q = 0. \quad (3)$$

(3) is call the indicial equation for m. It has solutions

$$m = \frac{-(p-1) \pm \sqrt{(p-1)^2 - 4q}}{2}$$

Let $\triangle = (p-1)^2 - 4q$ Then we have three cases just as we had for second order equations with constant coefficients.

Case 1. $\triangle > 0 \Rightarrow 2$ distinct roots $m_1, m_2 \Rightarrow$

$$y_h = c_1x^{m_1} + c_2x^{m_2}.$$

Case 2. $\triangle = 0 \Rightarrow$

$$m_1 = \frac{1-p}{2}.$$

To get a second solution let $y = u(x)x^{m_1}$, where $u(x)$ is a function to be determined. Equation (2) ⇒

$$xu'' + u' = 0.$$

Letting $v = u' \Rightarrow xv' + v = 0 \Rightarrow$

$$v' + \frac{v}{x} = 0.$$

The integrating factor for this equation is $e^{\int \frac{1}{x}dx} = e^{\ln x} = x \Rightarrow \frac{d}{dx}(xv) = 0 \Rightarrow xv = c_1 \Rightarrow v = \frac{c_1}{x}$
$u = \int v = c_1 \ln x + c_2$
$\Rightarrow$

$$y_h = x^{m_1}[c_1 \ln x + c_2].$$

Case 3. $\triangle < 0 \Rightarrow$ roots are complex conjugates, $m_1 = a + bi$ and $m_2 = a - bi$.
Thus

$$y_h = c_1 x^{m_1} + c_2 x^{m_2} = c_1 x^{a+bi} + c_2 x^{a-bi}$$

or

$$y_h = x^a[c_1 x^{bi} + c_2 x^{-bi}].$$

Now

$$x^{a+bi} = e^{(a+ib)\ln x} \text{ for } x > 0.$$

$\Rightarrow$

$$x^{bi} = e^{ib\ln x} = \cos(b\ln x) + i\sin(b\ln x)$$

$\Rightarrow$

$$y_h = x^a[A\cos(b\ln x) + B\sin(b\ln x)].$$

Example Solve

$$x^2y'' + 7xy' + 5y = 0$$

Here $p = 7$ and $q = 5$. The indicial equation (3) is for this example

$$m^2 + m(p-1) + q =$$
$$m^2 + 6m + 5 = (m+5)(m+1)$$

$\Rightarrow m = -5$ or $-1 \Rightarrow$

$$y = c_1 x^{-5} + c_2 x^{-1} = \frac{c_1}{x^5} + \frac{c_2}{x}.$$

Example Solve

$$x^2y'' + 3xy' + y = 0$$

Here $p = 3$ and $q = 1$ so the indicial equation is

$$m^2 + m(p-1) + q = m^2 + 2m + 1 = (m+1)^2$$

Thus $m = -1$ is a repeated root and the solution is

$$y_h = c_1 x^{-1} + c_2 x^{-1}\ln x$$

Example Solve

$$x^2y'' + 3xy' + 2y = 0$$

Here $p = 3$ and $q = 2$ so the indicial equation is

$$m^2 + m(p-1) + q = m^2 + 2m + 2$$

Thus

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = -1 \pm i$$

Thus we set $a = -1$ and $b = 1$ in the formula

$$y_h = x^a[A\cos(b\ln x) + B\sin(b\ln x)].$$

and get

$$y_h = x^{-1}[A\cos(\ln x) + B\sin(\ln x)]$$