Ma 221

Series Solutions of Differential Equations

Solution by Power Series

We shall now study ways of solving the second order differential equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

This equation has variable coefficients. In any interval where $a_2(x) \neq 0$, we can divide the equation by $a_2(x)$ to obtain $y'' + P(x)y' + Q(x)y = R(x)$. We shall consider only the homogeneous case

$$y'' + P(x)y' + Q(x)y = 0. \quad (1)$$

This equation will be solved by power series. It will turn out that near a point $x = a$

$$\begin{aligned} y &= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \\ &= \sum_{n=0}^{\infty} a_n(x-a)^n \end{aligned}$$

where $a_0, a_1, \ldots, a_n, \ldots$ are constants to be determined. This series is the Taylor series expansion of the solution y. Let us first begin with two definitions.

Definition 1. A function $f(x)$ is said to be analytic at $x = a$ if it can be expanded in a power series, in powers of $x - a$, which converges to $f(x)$ in an open interval containing $x = a$. This series is the Taylor series for $f(x)$.

Note: A necessary condition for $f(x)$ to be analytic is that $f(x)$ and its derivatives of all orders exist at $x = a$.
$f(x)$ analytic $\Rightarrow$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This series is called the Taylor series of $f(x)$ near $x = a$.
When the point $x = a = 0$, the series is called MacLaurin series. If $f(x)$ is not analytic at $x = a$, it is said to be singular or to have a singularity at $x = a$.

Examples:
1. $f(x) = \frac{1}{1-x} = (1-x)^{-1}$ is analytic at $x = 0$
$f'(x) = -(1-x)^{-2}(-1) = (1-x)^2$
$f''(x) = +2(1+x)^{-3} \quad f'''(x) = 3.2(1-x)^{-4}$
$f^{(n)}(x) = n!(1-x)^{-n-1}$ so that at $x = 0$ $f^{(n)}(0) = n!$
$\Rightarrow$ Taylor expansion for $\frac{1}{1-x}$ near $x = 0$ is
$f(x) = \sum_{n=0}^{\infty} \frac{n!}{n!}(x-0)^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$.
However, $\frac{1}{1-x}$ is not analytic at $x = 1$, since it approaches ∞ as $x \to 1$. $\Rightarrow$ no power series in powers of $(x-1)$.

2. $f(x) = x^{\frac{1}{n}} \quad n = 2, \ldots$ is not analytic at $x = 0$

$f'(x) = \frac{1}{n} x^{\frac{1}{n}-1} \quad n \geq 2 \Rightarrow \frac{1}{n} - 1 < 0 \Rightarrow f'(x)$ at $x = 0$ does not exist.

3. $f(x) = \frac{1}{x^2+1}$ analytic for all real x. However, for x complex, $x = \pm i$ is a singularity.

4. What are the singularities of $f(x) = \dfrac{x-1}{x^3 - 2x^2 + x}$?

$$f(x) = \frac{x-1}{x(x^2 - 2x + 1)} = \frac{x-1}{x(x-1)^2} = \frac{1}{x(x-1)}$$

$\Rightarrow x = 0$ and $x = 1$ are singularities.

Definition 2. The point $x = a$ is called an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0 \quad (1)$$

if both $P(x)$ and $Q(x)$ are analytic at $x = a$. If either $P(x)$ or $Q(x)$ is not analytic at $x = a$, then this point is called a singular point or singularity of the differential equation (1).

Example

$$(x^2 - 3x + 2)y'' + \sqrt{x}\,y' + x^2 y = 0$$

We rewrite the equation as

$$y'' + \frac{\sqrt{x}\ y'}{(x-2)(x-1)} + \frac{x^2}{(x-2)(x-1)}y = 0$$

Thus $P(x) = \frac{\sqrt{x}}{(x-2)(x-1)}$ and $Q(x) = \frac{x^2}{(x-2)(x-1)}$. Thus the equation has singularities at $x = 2,\ 1,$ and 0. $x = 0$ is a singular point because the derivative of $P(x)$ at 0 is not defined. All other points are ordinary points.

The theorem below gives conditions which insure the existence of a power series solution.

Theorem. If $x = a$ is an ordinary point of the differential equation (1), then $\exists$ two linearly independent power-series solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x-a)^n$$

These solutions will be valid in some interval containing $x = a$.

Method of Solution near an ordinary point.

Example Consider the differential equation

$$y'' + xy' + 2y = 0.$$

Here $P(x) = x$ and $Q(x) = 2$. They are both analytic $\forall\, x$, and in particular at $x = 0$. Hence by the above theorem $\exists$ two solutions of form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

(Here $a = 0$) The coefficients a_n are determined from the differential equation as follows.
Now

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

.
The differential equation ⇒

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + x \bullet \sum_{n=1}^{\infty} a_n n x^{n-1} + 2 \bullet \sum_{n=0}^{\infty} a_n x^n = 0.$$

or

$$2a_0 + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n(n+2)\ x^n = 0.\ (*)$$

We shall combine the coefficients of like powers of x in $(*)$ to get one power series. To do this we must put each term in the equation in the same form. This is accomplished by "shifting" the second series in $(*)$.

If we let $n = k - 2$ in the second series, $(*)$ becomes

$$2a_0 + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{k=3}^{\infty} a_{k-2}(k)x^{k-2} = 0.$$

Since n and k are "dummy" place keepers, we may replace them by m. Doing this yields

$$2a_0 + 2 \bullet 1 \bullet a_2 + \sum_{m=3}^{\infty} [a_m(m)(m-1) + a_{m-2}(m)]x^{m-2} = 0.$$

Remark. If $\sum_0^{\infty} a_n(x-a)^n = 0 \quad \forall\, x$ in some interval ⇒ $a_n = 0$ for $n = 0, 1, 2, ...$
Thus we have from the above equation

$$1.\quad 2(a_2 + a_0) = 0 \text{ or } a_2 = -a_0$$

$$2.\quad m = 3 \ \Rightarrow\ (3 \bullet 2a_3 + 3a_1) = 0 \text{ or } 2a_3 = -a_1 \ \Rightarrow\ a_3 = -\frac{1}{2}a_1$$

$$3.\ m = 4 \ \Rightarrow\ (4 \bullet 3a_4 + 4a_2) = 0 \text{ or } a_4 = -\frac{1}{3}a_2 = +\frac{1}{3}a_0$$

$$4.\ \Rightarrow [m(m-1)a_m + m a_{m-2}] = 0 \text{ or } a_m = -\frac{1}{m-1}a_{m-2} \ \text{ for } \ m \geq 3.$$

The expression in 4 is called the recurrence relation. Continuing we have for $m = 5$ and $m = 6$
$a_5 = -\frac{1}{4}a_3 = -\frac{1}{4}(-\frac{1}{2}a_1) = \frac{1}{4 \cdot 2}a_1$ and $a_6 = -\frac{1}{5}a_4 = -\frac{1}{5}(\frac{1}{3}a_0) = -\frac{1}{5 \cdot 3}a_0$
Hence the solution is

$$y = a_0 + a_1 + a_2x^2 + \cdots = a_0[1 - x^2 + \frac{1}{3}x^4 - \frac{1}{5 \bullet 3}x^6 + \cdots] + a_1[x - \frac{1}{2}x^3 + \frac{1}{4 \bullet 2}x^5 - \cdots].$$

It can be shown that in general

$$y = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{3 \cdot 5 \cdots (2n-1)} x^{2n} + a_1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2 \cdot 4 \cdots (2n-2)}$$

The above is the general solution of differential equation with two arbitrary constants a_0 and a_1.
Question: Where is the series solution valid? We shall use the ratio test to determine where the series converges.
Recall that if $\lim_{n\to\infty} \left| \frac{b_{n+k}}{b_n} \right| = L$ and $L < 1 \Rightarrow \sum b_n$ converges.
Recall that we have $a_m = -\frac{1}{m-1} a_{m-2}$.

$$\Rightarrow \lim_{m\to\infty} \left| \frac{a_{m+2}x^{m+2}}{a_m x^m} \right| = \lim_{m\to\infty} \left| \frac{-\frac{1}{m+1} a_m x^2}{a_m} \right| = \lim_{m\to\infty} \left| \frac{1}{m+1} x^2 \right| = 0$$

$\Rightarrow$ the series converges $\forall\, x$.

In general we have the following result about the convergence of a series solution.

Theorem. If $x = a$ is an ordinary point for the differential equation

$$y'' + P(x)y' + Q(x)y = 0,$$

then $\exists$ 2 L.I. series solutions of the form

$$y(x) = \sum_0^{\infty} a_n (x-a)^n.$$

These series converge at least $\forall$ values of x such that $|\ x-a\ | < R$, where R is the distance from the point $x = a$ to the nearest singular point of the D.E. in the complex plane.

Remark. The distance between $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ is $|\ z_1 - z_2\ | = [(a_1 - a_2)^2 + (b_1 - b_2)^2]^{\frac{1}{2}}$.

Example $(x^2 - 3x + 2)y'' + \sqrt{x}\ y' + x^2 y = 0$
$\Rightarrow$

$$y'' + \frac{\sqrt{x}}{(x-2)(x-1)} y' + \frac{x^2}{(x-2)(x-1)} y = 0$$

$x = 2,\ 1$ are singular points. Also $x = 0$ is a singular point due to the $\sqrt{x}$. $\exists$ a solution of form

$$y = \sum a_n (x-10)^n$$

about 10. By the theorem this converges $\forall x$ such that $|\ x - 10\ | < R$. Since $x = 2$ is the nearest singularity to $x = 10$, $R = |\ 10 - 2\ | = 8$.

Example Find the general solution near $x = 0$ of

$$y'' + xy = 0.$$

$y = \sum_0^{\infty} a_n x^n$ $y' = \sum_0^{\infty} n a_n x^{n-1} = \sum_1^{\infty} n a_n x^{n-1}$ and $y'' = \sum_1^{\infty} n(n-1) a_n x^{n-2} = \sum_2^{\infty} n(n-1) a_n x^{n-2}$.
The differential equation $\Rightarrow$

$$\sum_2^{\infty} a_n (n)(n-1) x^{n-2} + \sum_0^{\infty} a_n x^{n+1} = 0.$$

We must line up like powers of x. To do this both series must be of the same form. Consider

$$\sum_2^{\infty} a_n (n)(n-1) x^{n-2} = \sum_{k=-1}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1}$$

where we have let $n-2 = k+1 \Rightarrow n = k+3$ or $k = n-3$. When $n = 2 \Rightarrow k = -1$. The D.E. may now be written as

$$\sum_{k=-1}^{\infty}(k+3)(k+2)a_{k+3}x^{k+1} + \sum_{0}^{\infty} a_n x^{n+1} = 0$$

Replacing the "dummy" variables k and n by m, we have

$\Rightarrow$

$$2(1)a_2 + \sum_{0}^{\infty}\{(m+3)(m+2)a_{m+3} + a_m\}x^{m+1} = 0$$

$\Rightarrow a_2 = 0$ and

$$a_{m+3} = \frac{-a_m}{(m+3)(m+2)} \qquad k = 0,1,2,\ldots$$

$$m = 0 \quad \Rightarrow a_3 = \frac{-a_0}{3 \bullet 2} = \frac{-a_0}{6}$$

$$m = 1 \quad \Rightarrow a_4 = \frac{-a_1}{4 \bullet 3} = \frac{-a_1}{12} \qquad m = 2 \quad \Rightarrow a_5 = \frac{-a_2}{5 \bullet 4} = 0$$

$$m = 3 \quad \Rightarrow a_6 = \frac{-a_3}{6 \bullet 5} = \frac{a_0}{30 \bullet 6} = \frac{a_0}{180}$$

$$m = 4 \quad \Rightarrow a_7 = \frac{-a_4}{7 \bullet 6} = \frac{a_1}{7 \bullet 6 \bullet 12}$$

$$m = 5 \quad \Rightarrow a_8 = 0$$

$$m = 6 \quad \Rightarrow a_9 = \frac{-a_6}{9 \bullet 8} = \frac{-a_0}{9 \bullet 8(180)}$$

etc.

Hence

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0[1 - \frac{1}{6}x^3 + \frac{1}{560}x^6 - \frac{1}{72 \bullet 180}x^9 + \cdots] + a_1[x - \frac{1}{12}x^4 + \frac{1}{7 \bullet 6 \bullet 12}x^7 + \cdots].$$

Example Find the power series solution to

$$y'' - xy' + y = 0$$

near $x = 0$. Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first four nonzero terms the solution.

Solution:

Let

$$y = \sum_{n=0}^{\infty} a_n x^n. \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Plugging in gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

$\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Let $k = n-2$ and $n = k+2$

$\Rightarrow$

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0..$$

Replacing the "dummy" variabled k and n by m leads to

$\Rightarrow$

$$(2)(1)a_2 + a_0 + (3)(2)a_3 x + \sum_{m=2}^{\infty}[(m+2)(m+1)a_{m+2} - (m-1)a_m]x^m = 0.$$

$\Rightarrow$

$$a_2 = -\frac{a_0}{2}, \qquad a_3 = 0$$

and the Recurrence relation is:

$$a_{m+2} = \frac{a_m(m-1)}{(m+1)(m+2)}, \qquad m = 2, 3, \ldots$$

$$k = 2 : a_4 = \frac{a_2}{3(4)} = -\frac{a_0}{24}. \qquad k = 3 : a_5 = 0.$$

$\Rightarrow$

$$y = a_0\left(1 - \frac{x^2}{2} - \frac{x^4}{24} + \cdots\right) + a_1 x.$$

Example Find the first 5 nonzero terms of the power series solution about $x = 0$ for the DE:

$$(4 - x^2)y' + y = 0$$

Be sure to give the recurrence relation.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

The DE implies

$$(4 - x^2)\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$4\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Let $n - 1 = k$ in the first sum, that is $n = k + 1$ and let $j = n + 1$ in the second sum, that is $n = j - 1$ Then we have

$$4\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k - \sum_{j=2}^{\infty}(j-1)a_{j-1}x^j + \sum_{n=0}^{\infty} a_n x^n = 0$$

Since k, j, n are "dummy" place keepers we may replace them by m to get

$$4\sum_{m=0}^{\infty}(m+1)a_{m+1}x^m - \sum_{m=2}^{\infty}(m-1)a_{m-1}x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$4a_1 + a_0 + (8a_2 + a_1)x + \sum_{m=2}^{\infty}[4(m+1)a_{m+1} - (m-1)a_{m-1} + a_m]x^m = 0$$

This implies that

$$a_1 = -\frac{1}{4}a_0$$

$$a_2 = -\frac{1}{8}a_1 = \frac{1}{32}a_0$$

and the recurrence relation

$$a_{m+1} = \frac{(m-1)a_{m-1} - a_m}{4(m+1)} \quad m = 2, 3, \ldots$$

Therefore letting $m = 2$

$$a_3 = \frac{a_1 - a_2}{4(3)} = \frac{-\frac{1}{4} - \frac{1}{32}}{12}a_0 = -\frac{3}{128}a_0$$

Letting $m = 3$

$$a_4 = \frac{2a_2 - a_3}{4(4)} = \frac{\frac{1}{16} + \frac{3}{128}}{16}a_0 = \frac{11}{2048}a_0$$

Thus

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0\left[1 - \frac{1}{4}x + \frac{1}{32}x^2 - \frac{3}{128}x^3 + \frac{11}{2048}x^4 + \cdots\right]$$

Example This example is a video slide show. Slide Example

You will need Real Player to view this. To get it click on Real Player.

The material below is not covered in Ma 221 anymore.

Solution Near a Singular Point

Consider now the case where we seek the solution of

$$y'' + P(x)y' + Q(x)y = 0 \quad (1)$$

near a singular point of either P or Q , i.e. a point where either P or Q are not analytic. We shall use the Method of Frobinius. We cannot treat all singularities. We begin with a definition.

Definition. A point $x = a$ is said to be a regular singular point or a regular singularity of the D.E. (1) if

1. $x = a$ is a singular point of (1); and
2. $(x-a)P(x)$ and $(x-a)^2Q(x)$ are analytic at $x = a$.

Remark. Condition 2 $\Rightarrow$ $(x-a)P(x)$ and $(x-a)^2Q(x)$ have Taylor series at $x = a$. If $x = a$ is a singular point which is not regular, it is called an irregular singular point.

Ex. (1) $x^2y'' + pxy' + qy = 0$ Euler's equation. This may be rewritten as

$y'' + \frac{p}{x}y' + \frac{q}{x^2}y = 0$

$x = 0$ is a regular singular point since $xP(x) = x\frac{p}{x} = p$ and $x^2Q(x) = x^2\frac{q}{x^2} = q$

(2) $y'' + \frac{2}{x}y' + \frac{3}{x(x-1)^3}y = 0$

It is clear that $x = 0$ and $x = 1$ are singular points. We must examine each singularity separately to see if it is regular or irregular. Consider $x = 0$ first. Now $xP(x) = 2$ which is analytic near $x = 0$. also $x^2Q(x) = \frac{3x}{(x-1)^3}$ which is also analytic near $x = 0$. Therefore $x = 0$ is a regular singular point.

Now consider $x = 1$. Then $a = 1$ and

$(x-1)\,P(x) = \frac{2(x-1)}{x}$ which is analytic at $x = 1$

$(x-1)^2 Q(x) = \frac{3}{x(x-1)}$ which is not analytic at $x = 1$

$\Rightarrow x = 1$ is an irregular singular point.

Near a regular singular point we have

Theorem. At a regular singular point $x = a$ of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

there is at least one solution which possesses an expansion of the form

$y = (x-a)^{\alpha} \sum_{n=0}^{\infty} a_n (x-a)^n.$

In order to see how one solves equation (1) near a regular singular point $x = a$ in the easiest manner we shall assume $a = 0$. If $a \neq 0$, then let $t = x - a$ in the D.E. and solve in terms of t. $t = 0$ is then a regular singular point.

Now $y = x^{\alpha} \sum_0^{\infty} a_n x^n = \sum_0^{\infty} a_n x^{n+\alpha} \Rightarrow y' = \sum_0^{\infty} (n+\alpha)\, a_n x^{n+\alpha-1}$

and $y'' = \sum_0^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2}$

Now $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0 \Rightarrow$ that

$xP(x) = \sum_0^{\infty} p_n x^n$ and $x^2Q(x) = \sum_0^{\infty} q_n x^n$

The D.E. $y'' + P(x)y' + Q(x)y = 0$ may be multiplied by x^2 to get

$x^2y'' + x^2P(x)y' + x^2Q(x)y = 0$

$\Rightarrow x^2[\alpha(\alpha-1)a_0x^{\alpha-2} + (\alpha+1)(\alpha)a_1x^{\alpha-1} + \cdots]$

$+x[p_0 + p_1x + \ldots][\alpha a_0 x^{\alpha-1} + (\alpha+1)a_1x^{\alpha} + \cdots]$

$+[q_0 + q_1x + \ldots][a_0x^{\alpha} + a_1x^{\alpha+1} + \ldots] = 0$

$\Rightarrow [\alpha(\alpha-1)a_0x^{\alpha} + (\alpha+1)\alpha a_1^{\alpha+1} + \cdots]$

$+[\alpha p_0 a_0 x^{\alpha} + \alpha p_1 x^{\alpha+1} a_0 + p_0(\alpha+1)a_1x^{\alpha+1} + \cdots]$

$+[q_0a_0x^{\alpha} + q_0a_1x^{\alpha+1} + q_1a_0x^{\alpha+1} + \cdots] = 0$

Setting the coefficients of x^{α} equal to $0 \Rightarrow \alpha(\alpha-1)a_0 + \alpha p_0 a_0 + q_0 a_0 = 0$

$\Rightarrow \alpha(\alpha-1) + \alpha p_0 + q_0 = 0$ or

$$\alpha^2 + (p_0 - 1)\alpha + q_0 = 0 \qquad (2)$$

Equation (2) is called the indicial equation. This result is not surprising in light of the results we got for Euler's equation. Therefore if α is a root of (2) $\Rightarrow y = \sum a_n x^{\alpha+n}$ is a solution of (1) for this α. The $a_n's$ are determined from the D.E.

Remarks: Since $xP(x) = \sum p_n x^n$ and $x^2Q(x) = \sum q_n x^n$, p_0 and q_o are the first terms in the Taylor expansions of $xP(x)$ and $x^2Q(x)$. Thus

$p_0 = lim_{x\to 0} xP(x)$ and $q_0 = lim_{x\to 0} x^2P(x)$

Ex. Find a series solution of the D.E.

$9x^2\, y'' + (x+2)y = 0$ near $x = 0$

We rewrite the equation as $y'' + \frac{(x+2)}{9x^2}\, y = 0$

$P(x) = 0$ $Q(x) = \frac{(x+2)}{9x^2}$ so $x = 0$ is regular singular point.

$xP(x) = 0 = \sum p_n x^n$ so $p_n = 0 \Rightarrow p_0 = 0$

$x^2Q(x) = \frac{x+2}{9} = \frac{2}{9} + \frac{x}{9} = \sum q_n x^n \Rightarrow q_0 = lim_{x\to 0}\left(\frac{2}{9} + \frac{x}{9}\right) = \frac{2}{9}$

Therefore equation (2) for α becomes

$\alpha^2 - \alpha + \frac{2}{9} = 0$ or $(\alpha - \frac{2}{3})(\alpha - \frac{1}{3}) = 0$ and therefore $\alpha = \frac{2}{3}$ or $\alpha = \frac{1}{3}$.

$\Rightarrow$ solutions of the form $y = x^{\frac{1}{3}} \sum_0^{\infty} a_n x^n$ and $y = x^{\frac{2}{3}} \sum_0^{\infty} b_n x^n$.

Consider the case $\alpha = \frac{1}{3}$ Since $y = \sum_0^{\infty} a_n x^{n+\frac{1}{3}}$

$y' = \sum_0^{\infty} (n+\frac{1}{3}) a_n x^{n-\frac{2}{3}}$ and $y'' = \sum_0^{\infty} (n+\frac{1}{3})(n-\frac{2}{3}) a_n x^{n-\frac{5}{3}}$

D.E. $9x^2\, y'' + (x+2)y = 0 \Rightarrow$

$9\sum_0^{\infty} (n+\frac{1}{3})\,(n-\frac{2}{3})\, a_n x^{n+\frac{1}{3}} + x\sum_0^{\infty} a_n x^{n+\frac{1}{3}} + 2\sum_0^{\infty} a_n x^{n+\frac{1}{3}} = 0$

or $\sum_0^\infty \{9(n+\frac{1}{3})(n-\frac{2}{3})a_n + 2a_n\}\, x^{n+\frac{1}{3}} + \sum_0^\infty a_n x^{n+\frac{4}{3}} = 0$

$\Rightarrow \sum_0^\infty \{[(3n+1)(3n-2)+2]a_n\} x^{n+\frac{1}{3}} + \sum_{k=1}^\infty a_{k-1} x^{k+\frac{1}{3}} = 0$

Let $k+\frac{1}{3} = n+\frac{4}{3} \Rightarrow k = n+1 \Rightarrow$

$\sum_0^\infty \{[9n^2-3n-2+2]a_n\} x^{n+\frac{1}{3}} + \sum_{k=1}^\infty a_{k-1} x^{k+\frac{1}{3}} = 0.$

Or $\sum_1^\infty \{3m(3m-1)a_m + a_{m-1}\}\, x^{m+\frac{1}{3}} = 0 \quad \Rightarrow a_m = \frac{-a_{m-1}}{3m(3m-1)}$

$m = 1 \Rightarrow a_1 = \frac{a_0}{3\cdot 2} \qquad m = 2 \Rightarrow a_2 = -\frac{a_1}{6\cdot 5} = +\frac{a_0}{6\cdot 5\cdot 3\cdot 2} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6}$

$m = 3 \Rightarrow a_3 = -\frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$

Therefore one solution is $y_1 = a_o x^{\frac{1}{3}}(1 - \frac{x}{3\cdot 2} + \frac{x^2}{2\cdot 3\cdot 5\cdot 6} - \frac{x^3}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots)$

For $\alpha = \frac{2}{3}$ one gets

$y_2 = b_o x^{\frac{2}{3}}\,(1 - \frac{x}{3\cdot 4} + \frac{x^2}{3\cdot 4\cdot 6\cdot 7} - \frac{x^3}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} + \cdots)$

For the method of Frobinius we have

Theorem. If the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

has a regular singularity at $x = 0$ and if the roots α_1 and α_2 of the indicial equation are distinct and do not differ by an integer, then there are two linearly independent solutions of the form

$y_1(x) = x^{\alpha_1} \sum_0^\infty a_n x^n \quad y_2(x) = x^{\alpha_1} \sum_0^\infty b_n x^n$