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Ma 221 Exam IA Solutions 15S

1 [20 **pts**.] Solve the initial value problem.

$$\frac{dy}{dx} = \frac{2y}{x} + x^2\cos x \qquad y(\pi) = 2\pi^2$$

Solution: The differential equation is linear. We rewrite it in standard from.

$$\frac{dy}{dx} - \frac{2}{x}y = x^2\cos x$$

Compute the integrating factor.

$$p(x) = -\frac{2}{x}$$
$$\int p\,dx = -\int \frac{2}{x}dx = -2\ln x = \ln x^{-2}$$
$$\mu = e^{\int p\,dx} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiply by the integrating factor, identify the left side as a derivative and integrate.

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3} = \cos x$$
$$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = \cos x$$
$$\frac{1}{x^2}y = \int \cos x\,dx$$
$$= \sin x + C$$
$$y = x^2\sin x + Cx^2$$

We use the initial condition to evaluate the constant.

$$y(\pi) = 2\pi^2 = C\pi^2$$
$$C = 2$$

The solution is

$$y = x^2(\sin x + 2).$$

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2 [20 **pts**.] Solve the initial value problem.

$$2xy^3dx - (1 - x^2)dy = 0 \qquad y(0) = 1$$

Solution: The differential equation is separable. We rewrite it and integrate.

$$2xy^3dx = (1 - x^2)dy$$

$$\frac{2x}{1 - x^2}dx = \frac{1}{y^3}dy$$

$$\int \frac{2x}{1 - x^2}dx = \int \frac{1}{y^3}dy$$

$$-\ln|1 - x^2| = \frac{1}{-2y^2} + C$$

We use the initial condition to evaluate the constant.

$$y(0) = 1$$

$$-\ln 1 = 0 = \frac{1}{-2} + C$$

$$C = \frac{1}{2}$$

An implicit solution is

$$\frac{2}{y^2} = \frac{1}{2} + \ln|1 - x^2|.$$

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3 [35 **points**] Consider the differential equation

$$\left(3x^2y\right)dx + \left(3x^3 + 3\right)dy = 0$$

a. Show that the differential equation is not exact.

Solution: We let $M(x,y) = 3x^2y$ and $N(x,y) = \left(3x^3 + 3\right)$. Then $\frac{\partial M}{dy} = 3x^2$ and $\frac{\partial N}{dx} = 9x^2$. Since these are not equal, the differntial equation is not exact.

b. Find a value of n, such that multplying the equation by y^n results in an exact differential equation.

Solution: Multiplication by y^n gives this differential equation.

$$\left(3x^2y^{n+1}\right)dx + \left(3x^3y^n + 3y^n\right)dy = 0$$

Now, we have $M(x,y) = 3x^2y^{n+1}$ and $N(x,y) = 3x^3y^n + 3y^n$. The partial derivatives are

$$\frac{\partial M}{dy} = 3(n+1)x^2y^n$$
$$\frac{\partial N}{dx} = 9x^2y^n.$$

For the d.e. to be exact, these must be equal..

$$\begin{aligned} 3(n+1)x^2y^n &= 9x^2y^n \\ 3(n+1) &= 9 \\ n+1 &= 3 \\ n &= 2 \end{aligned}$$

c. The differential equation

$$\left(3x^2y^2 + 2x\right)dx + \left(2x^3y + 3y^2\right)dy = 0$$

is exact. Find a solution.

Solution:

We find the function, $F(x,y)$ for which the left side is the total differential. I.e.

$$\begin{aligned} F_x &= \left(3x^2y^2 + 2x\right) \\ F_y &= \left(2x^3y + 3y^2\right) \end{aligned}$$

Integrating the first equation gives

$$\begin{aligned} F &= \int\left(3x^2y^2 + 2x\right)\partial x \\ &= x^3y^2 + x^2 + g(y) \end{aligned}$$

Now, we match the second equation.

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$$\begin{aligned} F_y &= 2x^3y + g'(y) = \left(2x^3y + 3y^2\right) \\ g'(y) &= 3y^2 \\ g(y) &= y^3 \\ F &= x^3y^2 + x^2 + y^3. \end{aligned}$$

The solution is

$$F = x^3y^2 + x^2 + y^3 = C$$

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4 [25 **pts**.] Solve

$$\frac{dy}{dt} + 2y = y^2$$

Solution: This is a Bernoulli equation - in standard form. First a little algebra.

$$y^{-2}\frac{dy}{dt} + 2y^{-1} = 1$$

Let $z = y^{-1}$. Then $\frac{dz}{dt} = -y^{-2}\frac{dy}{dt}$. Substitution into the d.e. gives

$$-\frac{dz}{dt} + 2z = 1$$
$$\frac{dz}{dt} - 2z = -1$$

The integrating factor is

$$\mu = e^{\int -2dtx} = e^{-2t}.$$

Multiply by the integrating factor and integrate.

$$e^{-2t}\frac{dz}{dt} - 2e^{-2t}z = -e^{-2t}$$
$$\frac{d}{dz}\left(e^{-2t}z\right) = -e^{-2t}$$
$$e^{-2t}z = \frac{1}{2}e^{-2t} + C$$
$$z = \frac{1}{2} + Ce^{2t}$$

Remember to return to the original variable. The implicit solution is

$$\frac{1}{y} = \frac{1}{2} + Ce^{2t}$$

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Table of Integrals

$\int \sec^2 t\,dt = \tan t + C$
$\int \frac{\sec^2 t}{\tan t} dt = \ln(\tan t) + C$
$\int \tan t\,dt = \ln(\sec t) + C$
$\int te^{at}dt = \frac{1}{a^2}e^{at}(at - 1) + C$
$\int t^2e^{at}dt = \frac{1}{a^3}e^{at}\left(a^2t^2 - 2at + 2\right) + C$
$\int \cos^2 t\,dt = \frac{1}{2}t + \frac{1}{4}\sin 2t + C$
$\int \cos^3 t\,dt = \frac{1}{3}\cos^2 t \sin t + \frac{2}{3}\sin t + C$
$\int \sin^2 t\,dt = \frac{1}{2}t - \frac{1}{4}\pi - \frac{1}{4}\sin 2t + C$
$\int \sin^3 t\,dt = \frac{1}{12}\cos 3t - \frac{3}{4}\cos t + C$