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Ma 221 Exam IB Solutions 15S

1 [20 **pts**.] Solve the initial value problem.

$$\frac{dy}{dx} = \frac{3y}{x} + x^3\cos x \qquad y(\pi) = 2\pi^3$$

Solution: The differential equation is linear. We rewrite it in standard from.

$$\frac{dy}{dx} - \frac{3}{x}y = x^3\cos x$$

Compute the integrating factor.

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ \int p\,dx &= -\int \frac{3}{x}dx = -3\ln x = \ln x^{-3} \\ \mu &= e^{\int p\,dx} = e^{\ln x^{-3}} = x^{-3} = \frac{1}{x^3} \end{aligned}$$

Multiply by the integrating factor, identify the left side as a derivative and integrate.

$$\begin{aligned} \frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4} &= \cos x \\ \frac{d}{dx}\left(\frac{1}{x^3}y\right) &= \cos x \\ \frac{1}{x^3}y &= \int \cos x\,dx \\ &= \sin x + C \\ y &= x^3\sin x + Cx^3 \end{aligned}$$

We use the initial condition to evaluate the constant.

$$\begin{aligned} y(\pi) &= 2\pi^3 = C\pi^3 \\ C &= 2 \end{aligned}$$

The solution is

$$y = x^3(\sin x + 2).$$

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2 [20 **pts**.] Solve the initial value problem.

$$2x^3ydy - \left(1 - y^2\right)dx = 0 \qquad y(1) = 0.$$

Solution: The differential equation is separable. We rewrite it and integrate.

$$2x^3ydy = \left(1 - y^2\right)dx$$

$$\frac{2y}{1 - y^2}dy = \frac{1}{x^3}dx$$

$$\int \frac{2y}{1 - y^2}dy = \int \frac{1}{x^3}dx$$

$$-\ln\left|1 - y^2\right| = \frac{-2}{x^2} + C$$

We use the initial condition to evaluate the constant.

$$y(1) = 0$$

$$-\ln 1 = 0 = -2 + C$$

$$C = 2$$

An implicit solution is

$$\frac{2}{x^2} = 2 + \ln\left|1 - y^2\right|.$$

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3 [35 **points**] Consider the differential equation

$$\left(3y^3+3\right)dx+\left(3xy^2\right)dy=0$$

a. Show that the differential equation is not exact.

Solution: We let $M(x,y)=3y^3+3$ and $N(x,y)=3xy^2$. Then $\frac{\partial M}{dy}=9y^2$ and $\frac{\partial N}{dx}=3y^2$. Since these are not equal, the differntial equation is not exact.

b. Find a value of n, such that multplying the equation by x^n results in an exact differential equation.

Solution: Multiplication by x^n gives this differential equation.

$$\left(3x^ny^3+3x^n\right)dx+\left(3x^{n+1}y^2\right)dy=0$$

Now, we have $M(x,y)=3x^ny^3+3x^n$ and $N(x,y)=3x^{n+1}y^2$. The partial derivatives are

$$\frac{\partial M}{dy}=9x^ny^2$$
$$\frac{\partial N}{dx}=3(n+1)x^ny^2.$$

For the d.e. to be exact, these must be equal..

$$9x^ny^2=3(n+1)x^ny^2$$
$$3(n+1)=9$$
$$n+1=3$$
$$n=2$$

c. The differential equation

$$\left(2xy^3+3x^2\right)dx+\left(3x^2y^2+2y\right)dy=0$$

is exact. Find a solution.

Solution:

We find the function, $F(x,y)$ for which the left side is the total differential. I.e.

$$F_x=\left(2xy^3+3x^2\right)$$
$$F_y=\left(3x^2y^2+2y\right)$$

Integrating the first equation gives

$$F=\int\left(2xy^3+3x^2\right)\partial x$$
$$=x^2y^3+x^3+g(y)$$

Now, we match the second equation.

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$$\begin{aligned} F_y &= 3x^2y^2 + g'(y) = \left(3x^2y^2 + 2y\right) \\ g'(y) &= 2y \\ g(y) &= y^2 \\ F &= x^2y^3 + x^3 + y^2. \end{aligned}$$

The solution is

$$F = x^2y^3 + x^3 + y^2 = C$$

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4 [25 **pts**.] Solve

$$\frac{dy}{dt} + 3y = y^3$$

Solution: This is a Bernoulli equation - in standard form. First a little algebra.

$$y^{-3}\frac{dy}{dt} + 3y^{-2} = 1$$

Let $z = y^{-2}$. Then $\frac{dz}{dt} = -2y^{-3}\frac{dy}{dt}$. Substitution into the d.e. gives

$$-\frac{1}{2}\frac{dz}{dt} + 3z = 1$$

$$\frac{dz}{dt} - 6z = -2$$

The integrating factor is

$$\mu = e^{\int -6dtx} = e^{-6t}.$$

Multiply by the integrating factor and integrate.

$$e^{-6t}\frac{dz}{dt} - 6e^{-6t}z = -2e^{-6t}$$

$$\frac{d}{dz}\left(e^{-6t}z\right) = -2e^{-6t}$$

$$e^{-6t}z = \int -2e^{-6t}dt$$

$$= \frac{1}{3}e^{-6t} + C$$

$$z = \frac{1}{3} + Ce^{6t}$$

Remember to return to the original variable. The implicit solution is

$$\frac{1}{y^2} = \frac{1}{3} + Ce^{6t}$$

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Table of Integrals

$\int \sec^2 t dt = \tan t + C$
$\int \frac{\sec^2 t}{\tan t} dt = \ln(\tan t) + C$
$\int \tan t dt = \ln(\sec t) + C$
$\int t e^{at} dt = \frac{1}{a^2} e^{at}(at - 1) + C$
$\int t^2 e^{at} dt = \frac{1}{a^3} e^{at}\left(a^2 t^2 - 2at + 2\right) + C$
$\int \cos^2 t dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + C$
$\int \cos^3 t dt = \frac{1}{3} \cos^2 t \sin t + \frac{2}{3} \sin t + C$
$\int \sin^2 t dt = \frac{1}{2} t - \frac{1}{4} \pi - \frac{1}{4} \sin 2t + C$
$\int \sin^3 t dt = \frac{1}{12} \cos 3t - \frac{3}{4} \cos t + C$