

**Ma 221****Exam II A Solutions****15S**

- 1.** (30 pts. total) Consider the differential equation

$$L[y] = y'' + 4y' + 4y = 12e^{-2x} + 8x^2$$

**1 a (6 pts.)** Find the solution of the corresponding homogeneous equation  $L[y] = 0$ .

Solution: The characteristic equation is

$$p(r) = r^2 + 4r + 4 = (r + 2)^2 = 0.$$

Thus  $r = -2, 4$  and

$$y_h = c_1 e^{-2x} + c_2 x e^{-2x}.$$

**1 b (20 pts.)** Find a particular solution of this equation.

Solution: We first find  $y_{p_1}$  for  $12e^{-2x}$ . Since  $p(-2) = 4 - 8 + 4 = 0$  we must go a bit further.

$$\begin{aligned} p'(r) &= 2r + 4 \\ p'(-2) &= -4 + 4 = 0 \\ p''(r) &= 2 \neq 0 \end{aligned}$$

So, we have

$$y_{p_1} = \frac{12x^2 e^{-2x}}{2} = 6x^2 e^{-2x}.$$

To find  $y_{p_2}$  corresponding to  $8x^2$  we let

$$y_{p_2} = Ax^2 + Bx + C$$

Then

$$\begin{aligned} y'_{p_2} &= 2Ax + B \\ y''_{p_2} &= 2A \end{aligned}$$

Plugging into the DE we have

$$2A + 4(2Ax + B) + 4(Ax^2 + Bx + C) = 8x^2$$

$$4A = 8$$

$$8A + 4B = 0$$

$$2A + 4B + 4C = 0$$

, Solution is:  $[A = 2, B = -4, C = 3]$

Thus

$$y_{p_2} = 2x^2 - 4x + 3$$

$$\begin{aligned} y_p &= y_{p_1} + y_{p_2} \\ &= 6x^2 e^{-2x} + 2x^2 - 4x + 3 \end{aligned}$$

**1 c (4 pts.)** Give a general solution of the equation

$$L[y] = y'' + 4y' + 4y = 12e^{-2x} + 8x^2$$

Solution:

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^{-2x} + c_2 x e^{-2x} + 6x^2 e^{-2x} + 2x^2 - 4x + 3$$

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**2 (25 pts)** Find a particular solution of the differential equation

$$L[y] = y'' - y' - 2y = 36te^{-t}$$

Solution: First, we check the homogeneous solution.

$$\begin{aligned} p(r) &= r^2 - r - 2 = (r - 2)(r + 1) \\ y_h &= c_1 e^{2t} + c_2 e^{-t} \end{aligned}$$

Hence, we look for a solution in the form

$$\begin{aligned} y_p(t) &= t(At + B)e^{-t} \\ &= At^2 e^{-t} + Bte^{-t} \\ y'_p(t) &= -At^2 e^{-t} + 2At e^{-t} - Bte^{-t} + Be^{-t} \\ y''_p(t) &= At^2 e^{-t} - 2At e^{-t} - 2Ate^{-t} + 2Ae^{-t} + Bte^{-t} - Be^{-t} - Be^{-t} \end{aligned}$$

Substituting into the DE yields

$$\begin{aligned} L[y_p] &= [A + A - 2A]t^2 e^{-t} + [(-4A + B) + (-2A + B) - 2B]te^{-t} + [(2A - 2B) - B]e^{-t} \\ &= [-6At + (2A - 3B)]e^{-t} = 36te^{-t} \end{aligned}$$

We equate coefficients.

$$\begin{aligned} te^{-t} : \quad -6A &= 36 \\ e^{-t} : \quad 2A - 3B &= 0 \end{aligned}$$

So  $A = -6$ ,  $B = -4$  and the solution is

$$y_p = [-6t^2 - 4t]e^{-t}.$$

**3 (25 pts.)** Solve the equation

$$y'' - 2y' + y = \frac{1}{x}e^x, \quad x > 0$$

Solution: We first find the homogeneous solution. The characteristic equation is

$$r^2 - 2r + 1 = (r - 1)^2 = 0$$

Thus  $r = 1$  is a double root so

$$y_h = c_1 e^x + c_2 x e^x$$

To find a particular solution we use the Method of Variation of Parameters and let  $y_1 = e^x$  and  $y_2 = xe^x$ 

$$y_p = v_1 e^x + v_2 x e^x$$

Then the two equations for  $v'_1$  and  $v'_2$ ,

$$y_1 v'_1 + y_2 v'_2 = 0$$

$$y'_1 v'_1 + y'_2 v'_2 = \frac{f}{a}$$

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become

$$\begin{aligned} e^x v'_1 + x e^x v'_2 &= 0 \\ e^x v'_1 + (x+1) e^x v'_2 &= \frac{1}{x} e^x \end{aligned}$$

or

$$\begin{aligned} v'_1 + x v'_2 &= 0 \\ v'_1 + (x+1) v'_2 &= \frac{1}{x} \end{aligned}$$

Subtract the first equation from the second to obtain

$$\begin{aligned} v'_2 &= \frac{1}{x} \\ v_2 &= \ln x + c_2 \end{aligned}$$

Then from the first equation we have

$$\begin{aligned} v'_1 + x \cdot \frac{1}{x} &= 0 \\ v'_1 &= -1 \\ v_1 &= -x + c_1 \end{aligned}$$

so

$$y = e^x v_1 + x e^x v_2 = (-x + c_1) e^x + (\ln x + c_2) x e^x.$$

**4 (20 pts.)** Solve

$$x^2 y'' + 5xy' + 5y = 0$$

Solution: This is a Cauchy-Euler equation. The indicial (auxiliary) equation for a solution of the form  $x^r$  is

$$\begin{aligned} r(r-1) + 5r + 5 &= r^2 + 4r + 5 = 0 \\ r^2 + 4r + 4 + 1 &= 0 \\ (r+2)^2 &= -1 \\ r+2 &= \pm i \end{aligned}$$

So the roots are  $r = -2 \pm i$ . Thus, since

$$\begin{aligned} x^{-2+i} &= x^{-2} x^i = x^{-2} (e^{\ln x})^i \\ &= x^{-2} e^{i \ln x} \\ &= x^{-2} [\cos(\ln x) + i \sin(\ln x)], \end{aligned}$$

the solution is

$$y = c_1 x^{-2} \cos(\ln x) + c_2 x^{-2} \sin(\ln x).$$

## Table of Integrals

$$\int \ln t dt = t(\ln t - 1) + C$$

$$\int t \ln t dt = \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2 + C$$

$$\int (\ln t)^2 dt = t(\ln^2 t - 2\ln t + 2) + C$$

$$\int \frac{\ln t}{t} dt = \frac{1}{2} \ln^2 t + C$$

$$\int \sec u du = \ln(\sec u + \tan u) + C$$

$$\int \tan u du = \int \frac{\sin u}{\cos u} du = -\ln(\cos u) + C$$

$$\int \sin^2 u du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$$

$$\int \cos^2 u du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$$

$$\int u \sin u du = \sin u - u \cos u + C$$

$$\int u \cos u du = \cos u + u \sin u + C$$

An Identity

$$\sin u \tan u = \frac{\sin^2 u}{\cos u} = \frac{1 - \cos^2 u}{\cos u} = \sec u - \cos u$$