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Ma 221 Exam II B Solutions 15S

1. (30 **pts**. **total**) Consider the differential equation

$$L[y] = y'' - 4y' + 4y = 12e^{2x} + 16x^2$$

1 a (6 **pts**.) Find the solution of the corresponding homogeneous equation $L[y] = 0$.

Solution: The characteristic equation is

$$p(r) = r^2 - 4r + 4 = (r-2)^2 = 0.$$

Thus $r = -2, 4$ and

$$y_h = c_1 e^{2x} + c_2 x e^{2x}.$$

1 b (20 **pts**.) Find a particular solution of this equation.

Solution: We first find y_{p_1} for $12e^{2x}$. Since $p(2) = 4 - 8 + 4 = 0$ we must go a bit further.

$$\begin{aligned} p'(r) &= 2r - 4 \\ p'(2) &= 4 - 4 = 0 \\ p''(r) &= 2 \neq 0 \end{aligned}$$

So, we have

$$y_{p_1} = \frac{12x^2 e^{2x}}{2} = 6x^2 e^{2x}.$$

To find y_{p_2} corresponding to $16x^2$ we let

$$y_{p_2} = Ax^2 + Bx + C$$

Then

$$\begin{aligned} y'_{p_2} &= 2Ax + B \\ y''_{p_2} &= 2A \end{aligned}$$

Plugging into the DE we have

$$2A - 4(2Ax + B) + 4\left(Ax^2 + Bx + C\right) = 16x^2$$

$$\begin{aligned} 4A &= 16 \\ -8A + 4B &= 0 \\ 2A - 4B + 4C &= 0 \end{aligned}$$

, Solution is: $[A = 4, B = 8, C = 6]$

Thus

$$\begin{aligned} y_{p_2} &= 4x^2 + 8x + 6 \\ y_p &= y_{p_1} + y_{p_2} \\ &= 6x^2 e^{2x} + 4x^2 + 8x + 6 \end{aligned}$$

1 c (4 **pts**.) Give a general solution of the equation

$$L[y] = y'' - 4y' + 4y = 12e^{2x} + 16x^2$$

Solution:

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^{-2x} + c_2 x e^{-2x} + 6x^2 e^{2x} + 4x^2 + 8x + 6$$

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2 (25 **pts**) Find a particular solution of the differential equation

$$L[y] = y'' + y' - 2y = 18te^t$$

Solution: First, we check the homogeneous solution.

$$p(r) = r^2 + r - 2 = (r+2)(r-1)$$
$$y_h = c_1 e^{-2t} + c_2 e^t$$

Hence, we look for a solution in the form

$$y_p(t) = t(At+B)e^t$$
$$= At^2 e^{-t} + Bte^t$$

$$y_p'(t) = At^2 e^t + 2Ate^t + Bte^t + Be^t$$
$$y_p''(t) = At^2 e^t + 2Ate^t + 2Ate^t + 2Ae^t + Bte^t + Be^t + Be^t$$

Substituting into the DE yields

$$L[y_p] = [A + A - 2A]t^2 e^t + [(4A+B) + (2A+B) - 2B]te^t + [(2A+2B) + B]e^t$$
$$= [6At + (2A+3B)]e^t = 18te^t$$

We equate coefficients.

$$te^t : \qquad 6A = 18$$
$$e^t : \qquad 2A + 3B = 0$$

So $A = 3$, $B = -2$ and the solution is

$$y_p = \left[3t^2 - 3t\right]e^t.$$

3 $\left(25\ \textbf{pts.}\right)$ Solve the equation

$$y'' - 2y' + y = \frac{3}{x}e^x \qquad , x > 0$$

Solution: We first find the homogeneous solution. The characteristic equation is

$$r^2 - 2r + 1 = (r-1)^2 = 0$$

Thus $r = 1$ is a double root so

$$y_h = c_1 e^x + c_2 xe^x$$

To find a particular solution we use the Method of Variation of Parameters and let $y_1 = e^x$ and $y_2 = xe^x$

$$y_p = v_1 e^x + v_2 xe^x$$

Then the two equations for v_1' and v_2' ,

$$y_1 v_1' + y_2 v_2' = 0$$
$$y_1' v_1' + y_2' v_2' = \frac{f}{a}$$

become

$$\begin{aligned} e^x v_1' + xe^x v_2' &= 0 \\ e^x v_1' + (x+1)e^x v_2' &= \frac{3}{x}e^x \end{aligned}$$

or

$$\begin{aligned} v_1' + xv_2' &= 0 \\ v_1' + (x+1)v_2' &= \frac{3}{x} \end{aligned}$$

Subtract the first equation from the second to obtain

$$\begin{aligned} v_2' &= \frac{3}{x} \\ v_2 &= 3\ln x + c_2 \\ &= \ln\left(x^3\right) + c_2 \end{aligned}$$

Then from the first equation we have

$$\begin{aligned} v_1' + x \cdot \frac{3}{x} &= 0 \\ v_1' &= -3 \\ v_1 &= -3x + c_1 \end{aligned}$$

so

$$y = e^x v_1 + xe^x v_2 = (-3x + c_1)e^x + \left[\ln\left(x^3\right) + c_2\right]xe^x.$$

4 $\left(20\ \textbf{pts.}\right)$ Solve

$$x^2 y'' - 3xy' + 5y = 0$$

Solution: This is a Cauchy-Euler equation. The indicial (auxiliary) equation for a solution of the form x^r is

$$\begin{aligned} r(r-1) - 3r + 5 = r^2 - 4r + 5 &= 0 \\ r^2 - 4r + 4 + 1 &= 0 \\ (r-2)^2 &= -1 \\ r - 2 &= \pm i \end{aligned}$$

So the roots are $r = 2 \pm i$. Thus, since

$$\begin{aligned} x^{2+i} = x^2 x^i &= x^2\left(e^{\ln x}\right)^i \\ &= x^2 e^{i\ln x} \\ &= x^2[\cos(\ln x) + i\sin(\ln x)], \end{aligned}$$

the solution is

$$y = c_1 x^2 \cos(\ln x) + c_2 x^2 \sin(\ln x).$$

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Table of Integrals

$$\int \ln t dt = t(\ln t - 1) + C$$

$$\int t \ln t = \frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 + C$$

$$\int (\ln t)^2 dt = t\left(\ln^2 t - 2\ln t + 2\right) + C$$

$$\int \frac{\ln t}{t} dt = \frac{1}{2} \ln^2 t + C$$

$$\int \sec u du = \ln(\sec u + \tan u) + C$$

$$\int \tan u du = \int \frac{\sin u}{\cos u} du = -\ln(\cos u) + C$$

$$\int \sin^2 u du = \frac{1}{2} u - \frac{1}{4} \sin 2u + C$$

$$\int \cos^2 u du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C$$

$$\int u \sin u du = \sin u - u \cos u + C$$

$$\int u \cos u du = \cos u + u \sin u + C$$

An Identity

$$\sin u \tan u = \frac{\sin^2 u}{\cos u} = \frac{1 - \cos^2 u}{\cos u} = \sec u - \cos u$$