Ma 221 Exam IIIB Solutions 15S

1a (10 **pts**.)

$$f(t) = \begin{cases} 0 & if \quad 0 \le t < 3 \\ e^{2t} & if \quad 3 \le t \end{cases}$$

Use the definition of the Laplace transform to determine the Laplace transform of $f(t)$.

Solution:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)dt = \int_3^\infty e^{-st} e^{2t} dt \\ &= \int_3^\infty e^{(2-s)t} dt = \lim_{L\to\infty} \int_3^L e^{(2-s)t} dt \\ &= \lim_{L\to\infty} \frac{e^{(2-s)t}}{2-s} \Big|_{t=3}^{t=L} = \frac{(e^{(2-s)s} - e^{-1})}{2-s} \\ &= \frac{-e^{-1}}{2-s} = \frac{e^{-1}}{s-2} \end{aligned}$$

For the limit to exist, the coefficient of t must be negative. So the domain (region of validity) is $2 - s < 0$ which may be rwwritten as $s > 2$.

1b (15 **pts**.) Determine

$$\mathcal{L}^{-1}\left\{ \frac{3s+3}{s^2+6s+13} \right\}$$

Solution:

$$\begin{aligned} \frac{3s+3}{s^2+6s+13} &= \frac{3s+3}{s^2+6s+9+4} = \frac{3s+3}{(s+3)^2+2^2} \\ &= \frac{3(s+3)}{(s+3)^2+2^2} + \frac{-6}{(s+3)^2+2^2} \\ &= 3 \cdot \frac{(s+3)}{(s+3)^2+2^2} - 3 \cdot \frac{2}{(s+3)^2+2^2} \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}^{-1}\left\{ \frac{3s+3}{s^2+6s+13} \right\} &= 3\mathcal{L}^{-1}\left\{ \frac{(s+3)}{(s+3)^2+2^2} \right\} - 3\mathcal{L}^{-1}\left\{ \frac{2}{(s+3)^2+2^2} \right\} \\ &= 3e^{-3t}\cos 2t - 3e^{-3t}\sin 2t \end{aligned}$$

2a $\left(15 \ \textbf{pts.}\right)$ Consider the initial value problem

$$y'' + 3y' + 2y = 12e^{-2t} \quad y(0) = 2 \ \ y'(0) = -8$$

Let $Y(s) = \mathcal{L}\{y\}(s)$. Use Laplace transforms to show that

$$Y(s) = \frac{12}{(s+1)(s+2)^2} + \frac{2s-2}{(s+1)(s+2)}.$$

Solution: Taking the Laplace transform of both sides of the DE we have

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 12\mathcal{L}\{e^{-2t}\}$$

or letting $Y(s) = \mathcal{L}\{y\}(s)$

$$s^2Y(s) - sy(0) - y'(0) + 3\{Y(s) - y(0)\} + 2Y(s) = \frac{12}{s+2}$$

Using the given initial conditions we have

$$(s^2 + 3s + 2)Y(s) - 2s + 2 = \frac{12}{s+2}$$

Thus

$$Y(s) = \frac{12}{(s+2)^2(s+1)} + \frac{2s-2}{(s+2)(s+1)}$$

2b $\left(15 \ \textbf{pts.}\right)$ Find the solution to the initial problem above, namely,

$$y'' + 3y' + 2y = 12e^{-2t} \quad y(0) = 2 \ \ y'(0) = -8$$

by finding

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{12}{(s+1)(s+2)^2} + \frac{2s-2}{(s+1)(s+2)}\right\}.$$

Solution:
In order to invert $Y(s)$ we need to do a partial fractions breakup of $Y(s)$.
The form is

$$\frac{12}{(s+1)(s+2)^2} + \frac{2s-2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}.$$

The remainder of the solution is not required. However, here are the details. Multiplication by the common denominator gives

$$12 + (2s - 2)(s + 2) = A(s + 2)^2 + B(s + 2)(s + 1) + C(s + 1)$$

Set $s = -1$

$$12 - 4 = 8 = A$$

Set $s = -2$

$$12 = -C$$

Equate the coefficient of s^2 on each side of the equation.

$$2 = A + B$$

$B = 2 - A = -6$. Now, combining everything, we have

$$\begin{aligned} Y(s) &= \frac{12}{(s+1)(s+2)^2} + \frac{2s-2}{(s+1)(s+2)} \\ &= \frac{8}{s+1} + \frac{-6}{s+2} + \frac{-12}{(s+2)^2} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{8}{s+1} + \frac{-6}{s+2} + \frac{-12}{(s+2)^2}\right\} \\ &= 8\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 6\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - 12\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} \\ &= 8e^{-t} - 6e^{-2t} - 12te^{-2t} \end{aligned}$$

3 $\left(25\ \textbf{pts}.\right)$ Find the first 5 nonzero terms of the power series solution about $x = 0$ for the DE:

$$y'' - 2xy = 0$$

Be sure to give the recurrence relation.
Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

so

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

.
The differential equation ⇒

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Shifting the first series by letting $n - 2 = k$ or $n = k + 2$, we have

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Shifting the first series by letting $n + 1 = k$ or $n = k - 1$ we have

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - 2\sum_{k=1}^{\infty} a_{k-1} x^k = 0$$

Since the first series has one more term, we have

$$2a_2 + \sum_{k=1}^{\infty}[a_{k+2}(k+2)(k+1) - 2a_{k-1}]x^k = 0$$

Thus

$$a_2 = 0$$

and we have the recurrence relation

$$a_{k+2}(k+2)(k+1) - 2a_{k-1} = 0 \quad \text{for } k = 1,2,3,4,\ldots$$

or

$$a_{k+2} = \frac{2}{(k+2)(k+1)} a_{k-1} \quad \text{for } k = 1,2,3,4,\ldots$$

Therefore

$$
\begin{aligned}
a_3 &= \frac{2}{3 \cdot 2} a_0 \\
a_4 &= \frac{2}{4 \cdot 3} a_1 \\
a_5 &= \frac{2}{5 \cdot 4} a_2 = 0 \\
a_6 &= \frac{2}{6 \cdot 5} a_3 = \frac{4}{6 \cdot 5 \cdot 3 \cdot 2} a_0
\end{aligned}
$$

Thus

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0\left[1 + \frac{2}{3 \cdot 2}x^3 + \frac{4}{6 \cdot 5 \cdot 3 \cdot 2}x^6 + \ldots\right] + a_1\left[x + \frac{2}{4 \cdot 3}x^4 + \ldots\right]$$

4 (25 **pts**.) Find all eigenvalues (λ) and the corresponding eigenfunctions for the boundary value problem

$$y'' - 2y + \lambda y = 0 \qquad y'(0) = y'(\pi) = 0$$

Be sure to consider all values of λ.

Solution: This is an equation with constant coefficients. The characteristic equation is

$$r^2 - 2 + \lambda = 0$$

Thus

$$r = \sqrt{2 - \lambda}$$

There are 3 cases to consider: $2 - \lambda > 0$, $2 - \lambda = 0$, and $2 - \lambda < 0$.

Case I: $2 - \lambda > 0$, that is $\lambda < 2$. Let $2 - \lambda = \mu^2 \neq 0$. We have the two roots $r = \pm\upsilon$, and the solution

$$
\begin{aligned}
y(x) &= c_1 e^{\mu s} + c_2 e^{-\mu x} \\
y'(x) &= \mu(c_1 e^{\mu s} - c_2 e^{-\mu x})
\end{aligned}
$$

$$y'(0) = c_1 - c_2 = 0$$

so $c_1 = c_2$ and

$$y'(x) = \mu c_1(e^{\mu s} - e^{-\mu x})$$

Then

$$y'(\pi) = \mu c_1(e^{\pi\mu} - e^{-\pi\mu}) = 0$$

Since $e^{\pi\mu} - e^{-\pi\mu} \neq 0$ this implies that $c_1 = 0$ and therefore $c_2 = 0$ and the only solution for $\lambda < -3$ is the trivial solution $y = 0$. Hence there are no eigenvalues if $\lambda < -3$.

Case II: $\lambda = 2.$ Then $r = 0$ is a repeated root, and

$$y(x) = c_1 + c_2 x$$
$$y'(x) = c_2$$
$$y'(0) = c_2 = 0$$
$$y'(x) = 0$$
$$y'(2) = 0$$

Therefore c_1 may be any vallue and $\lambda = 2$ is an eigenvalue. Anticipating additional values in the next case, we write the eigenvalue and corresponding eigenfunction as

$$\lambda_0 = 2$$
$$y_0 = c_0.$$

Case III. $2 - \lambda < 0,$ that is $\lambda > 2.$ Let $2 - \lambda = -\mu^2 \neq 0.$ We have the complex roots $r = \pm\mu i$ and

$$y(x) = [c_1 \cos(\mu x) + c_2 \sin(\mu x)]$$
$$y'(x) = \mu[-c_1 \sin(\mu x) + c_2 \cos(\mu x)]$$
$$y'(0) = \mu c_2 = 0$$

so $c_2 = 0.$
Thus

$$y'(x) = -\mu c_1 \sin(\mu x)$$
$$y'(\pi) = -\mu c_1 \sin(\pi\mu)$$

So, for a non-zero solution, we must have

$$\sin(\pi\mu) = 0$$

Thus $\pi\mu$ must be an integral multiple of π..

$$\pi\mu = n\pi \qquad n = 1,2,3,\ldots$$

or

$$\mu_n = n \qquad n = 1,2,3,\ldots$$

And finally

$$\lambda_n = 2 + \mu^2 = 2 + n^2 \quad n = 1,2,3,\ldots$$

are eigenvalues with corresponding eigenfunctions

$$y_n(x) = c_n \cos(nx) \quad n = 1,2,3,\ldots.$$

Finally, we may combine cases II and III to have the eigenvalues and eigenfunctions

$$\lambda_n = 2 + n^2 \qquad n = 0,1,2,3,\ldots$$
$$y_n(x) = c_n \cos(nx) \quad n = 0,1,2,3,\ldots.$$

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s) = \hat{f}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		