May 2015

Print Name: ________________________________ **Lecture Section**: _________

Lecturer___________________

1.Solve the following initial value problems.

(a) (10 pts)

$$x\frac{dv}{dx} = \frac{1+4v^2}{4v}, \quad v(1) = 0.$$

Solution: The d.e. is separable. We rewrite and integrate.

$$\frac{4v}{1+4v^2}dv = \frac{1}{x}dx$$

$$\int \frac{4v}{1+4v^2}dv = \int \frac{1}{x}dx$$

$$\frac{1}{2}\ln(1+4v^2) = \ln x + c$$

From the initial condition, we obtain $c = 0$. So the implicit solution can be written in vaious ways according to taste. Here are a few.

$$\frac{1}{2}\ln(1+4v^2) = \ln x$$

$$\ln(1+4v^2) = 2\ln x = \ln(x^2)$$

$$(1+4v^2) = x^2$$

(b) (10 pts)

$$\frac{dy}{dx} + 4xy = 8x, \qquad y(0) = 0$$

Solution: This d.e. is linear. The integrating factor is $\mu = \exp\left(\int 4x dx\right) = \exp(2x^2)$. We multiply by the integrating factor, gather terms and integrate.

$$e^{2x^2}\left(\frac{dy}{dx} + 4xy\right) = 8xe^{2x^2}$$

$$\frac{d}{dx}\left(e^{2x^2}y\right) = 8xe^{2x^2}$$

$$e^{2x^2}y = \int 8xe^{2x^2}dx = 2e^{2x^2} + c$$

Applying the initial condition gives $0 = 2 + c$, so $c = -2$. The explicit solution is

$$y = 2 - 2e^{-2x^2}.$$

1 (c) (10 pts)

$$(2xy - 3x^2)dx + \left(x^2 - \frac{2}{y^3}\right)dy = 0, \qquad y(1) = -1$$

Solution. The equation is not separable. The terms if x^2 and y^3 show that it is not linear in either variable. We test for an exact d.e.

$$M = (2xy - 3x^2) \quad \frac{\partial M}{\partial y} = 2x$$

$$N = \left(x^2 - \frac{2}{y^3}\right) \quad \frac{\partial N}{\partial x} = 2x$$

The cross partial deerivatives are equal, so the d.e. is exact. We look for $F(x,y)$ such that $F_x = M$ and $F_y = N$. We start with the first equation.

$$F_x = M = 2xy - 3x^2$$

$$F = \int (2xy - 3x^2)\partial x$$

$$= x^2y - x^3 + g(y)$$

Now, for this F, we match the second condition.

$$F_y = x^2 + g'(y) = N = \left(x^2 - \frac{2}{y^3}\right)$$

$$g'(y) = N = -\frac{2}{y^3}$$

$$g(y) = \frac{1}{y^2}$$

$$F(x,y) = x^2y - x^3 + \frac{1}{y^2}$$

Hence the solution to the differential equation is $F(x,y) = x^2y - x^3 + \frac{1}{y^2} = c$. The initial condition yields $c = -1 - 1 + 1 = -1$. Hence, the implicit solution is

$$x^2y - x^3 + \frac{1}{y^2} = -1.$$

2. (a) (8 pts) Find a general solution of

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0, \qquad \text{for} \quad -\infty < x < \infty.$$

Solution: This is a homogeneous linear equation with constnt coefficients. The characteristic equation is

$$\begin{aligned} r^2 + 2r + 10 &= 0 \\ (r^2 + 2r + 1) + 9 &= 0 \\ (r+1)^2 &= -9 \\ r + 1 &= \pm 3i \\ r &= -1 \pm 3i \end{aligned}$$

A general solution is

$$y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x).$$

2(b) (8 pts.) Find a general solution of

$$t^2\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + 5y = 0, \qquad \text{for } t > 0.$$

Solution: This homogeneous linear equation is of the Cauchy-Euler (equidimensional) type. The indicial equation is

$$\begin{aligned} r(r-1) + 3r + 5 &= 0 \\ r^2 + 2r + 5 &= 0 \end{aligned}$$

$$\begin{aligned} (r^2 + 2r + 1) + 4 &= 0 \\ (r+1)^2 &= -4 \\ r + 1 &= \pm 2i \\ r &= -1 \pm 2i \end{aligned}$$

Since $x^{-1+2i} = x^{-1}x^{2i} = x^{-1}(e^{\ln x})^{2i} = x^{-1}(e^{2i\ln x})$, we may write a general solution as

$$y = \frac{1}{x}[c_1\cos(2\ln x) + c_2\sin(2\ln x)].$$

2(c) (14 pts.) Use the method of undetermined coefficients to find a general solution of

$$L[y] = y'' + 5y' + 4y = e^{-t} + 4te^{-2t}.$$

Solution: First, we check the homogeneous equation. The characteristic equation is $p(r) = r^2 + 5r + 4 = (r+4)(r+1) = 1.$ Roots are -1 and -4, so $y_h = c_1 e^{-4t} + c_2 e^{-t}$. Since $p(-1) = 0$, the solution to $L[y] = e^{-t}$ is

$$y_{p_1} = \frac{1}{p'(-1)} te^{-t} = \frac{1}{3} te^{-t}.$$

Now, for $L[y] = 4te^{-2t}$, we set $y_{p_2} = (At+B)e^{-2t}$ and substitute into the d.e.

$$\begin{aligned}
y &= (At+B)e^{-2t} \\
y' &= Ae^{-2t} - 2(At+B)e^{-2t} = [-2At + (A-2B)]e^{-2t} \\
y'' &= -2Ae^{-2t} - 2Ae^{-2t} + 4(At+B)e^{-2t} = [4At + (4B-4A)]e^{-2t} \\
L[y] &= \{[4At + (4B-4A)] + 5[-2At + (A-2B)] + 4(At+B)\}e^{-2t} \\
&= [-2At + (A-2B)]e^{-2t} = 4te^{-2t} \\
-2A &= 4 \quad \Rightarrow A = -2 \\
A - 2B &= 0 \quad \Rightarrow B = -1 \\
y_{p_2} &= (-2t-1)e^{-2t}
\end{aligned}$$

Fianlly, we combine everything to obtain a general solution.

$$\begin{aligned}
y &= y_h + y_{p_1} + y_{p_2} \\
&= c_1 e^{-4t} + c_2 e^{-t} + \frac{1}{3} te^{-t} - (2t+1)e^{-2t}.
\end{aligned}$$

3 (a) (15 pts.) Consider the differential equation

$$L[y] = t^2 \frac{d^2 y}{dt^2} + 4t\frac{dy}{dt} + 2y = 4\ln t, \qquad for \;\; t > 1.$$

Solutions to the homogeneous equation are $y_1(t) = \frac{1}{t}$ and $y_2(t) = \frac{1}{t^2}$.

(i) Use the Wronskian, $W[y_1, y_2](t)$, to verify that y_1 and y_2 are linearly independent solutions on the interval $t > 1$.

Solution:

$$\begin{aligned}
W[y_1, y_2] &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \frac{1}{t} & \frac{1}{t^2} \\ \frac{-1}{t^2} & \frac{-2}{t^3} \end{vmatrix} nthe \\
&= \left(\frac{1}{t}\right)\left(\frac{-2}{t^3}\right) - \left(\frac{1}{t^2}\right)\left(\frac{-1}{t^2}\right) \\
&= \frac{-2+1}{t^4} = \frac{-1}{t^4}.
\end{aligned}$$

Since the Wronskian is never zero on the interval $(0, \infty)$, the functions are linearly independent.

(ii) Find a particular solution, $y_p(t)$ satisfying

$$L[y] = t^2\frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + 2y = 4\ln t.$$

We use the method of variation of parameters to seek a solution of the form $y = v_1y_1 + v_2y_2$ using the given solutions.

$$y_1v_1' + y_2v_2' = \frac{1}{t}v_1' + \frac{1}{t^2}v_2' = 0$$
$$y_1'v_1' + y_2'v_2' = \frac{-1}{t^2}v_1' + \frac{-2}{t^3}v_2' = \frac{4\ln t}{t^2}$$

Multiply the first equation by t and the second equation by t^2.

$$v_1' + \frac{1}{t}v_2' = 0$$
$$-v_1' + \frac{-2}{t}v_2' = 4\ln t$$

Add these and the rest is straight forward.

$$\begin{aligned}
\frac{-1}{t}v_2' &= 4\ln t\\
v_2' &= -4t\ln t\\
v_2 &= -4\left(\frac{1}{2}t^2\ln t - \frac{1}{4}t^2\right) + c_2\\
v_1' &= -\frac{1}{t}v_2' = 4\ln t\\
v_1 &= 4(t\ln t - t) + c_1\\
y_p &= 4(t\ln t - t)\frac{1}{t} - 4\left(\frac{1}{2}t^2\ln t - \frac{1}{4}t^2\right)\frac{1}{t^2}
\end{aligned}$$

(iii) Find a general solution to the equation.
Solution:

$$y = [c_1 + 4(t\ln t - t)]\frac{1}{t} + \left[c_2 - 4\left(\frac{1}{2}t^2\ln t - \frac{1}{4}t^2\right)\right]\frac{1}{t^2}.$$

3 (b) (15 pts.) Classify each of the following differential equations as linear or nonlinear. If nonlinear, identify all terms that make the equation nonlinear. (In all cases, consider y to be the dependent variable and t the independent variable.) Solution:

	Equation	Linear/nonlinear	Nonlinear terms
(i)	$y'' + 3y' + 4\sin(y) = 0$	nonlinear	$\sin(y)$
(ii)	$y'' + 3y' + 4y = \cos(4t)$	linear	
(iii)	$t^2y'' - ty' + y = \ln(t)/t$	linear	
(iv)	$e^{-t}dy + (t^2y - \sin t)dt = 0$	linear	
(v)	$y'' + 2yy' + 4y = 0$	nonlinear	yy’
(vi)	$ty' + t^3y^2 = 4t^2$	nonlinear	y^2

4. (a) (9 pts.) Find the inverse Laplace transform for

$$F(s) = \frac{2s+1}{(s+2)^3}$$

Solution:

$$F(s) = \frac{2s+1}{(s+2)^3} = \frac{2(s+2)-3}{(s+2)^3} = \frac{2}{(s+2)^2} + \frac{-3}{(s+2)^3}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-3}{(s+2)^3}\right\}$$

$$= 2te^{2t} - \frac{3}{2}t^2e^{-2t}$$

4 (b) (6pts.) Determine the Laplace transform for $f(t) = t\sin(3t)$.
Solution:

$$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$$

$$\mathcal{L}\{t\sin 3t\} = (-1)\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = (-3)\frac{(-1)2s}{(s^2+9)^2}$$

$$= \frac{6s}{(s^2+9)^2}$$

4 (c) (15 pts.) Solve using Laplace Transforms:

$$L[y] = y'' + 2y' + 5y = te^{-2t}, \qquad y(0) = 0,\ y'(0) = 1.$$

Solution:

$$\begin{aligned}
\mathcal{L}\{y'' + 2y' + 5y\} &= \mathcal{L}\{te^{-2t}\} \\
s^2Y - sy(0) - y'(0) + 2[sY - y(0)] + 5Y &= \frac{1}{(s+2)^2} \\
(s^2 + 2s + 5)Y - 1 &= \frac{1}{(s+2)^2} \\
Y &= \frac{1}{(s+1)^2 + 2^2} + \frac{1}{\left[(s+1)^2 + 2^2\right](s+2)^2}
\end{aligned}$$

The first fraction on the right side is simple. We apply the partial fractions technique to the second.

$$\frac{1}{\left[(s+1)^2 + 2^2\right](s+2)^2} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{C(s+1) + 2D}{(s+1)^2 + 2^2}$$

Multiplication by the common denominator gives

$$1 = A(s+2)\left[(s+1)^2 + 2^2\right] + B\left[(s+1)^2 + 2^2\right] + [C(s+1) + 2D](s+2)^2.$$

Setting $s = -2$ gives B.

$$1 = B[1+4] \qquad \Rightarrow B = \frac{1}{5}$$

The coefficient of s^3 is easy to see.

$$0 = A + C \qquad \Rightarrow C = -A$$

Two more equations can be obtained from $s = 0$ (the constant term) and $s = -1$.

$$\begin{aligned}
1 &= A(2)(5) + B(5) + (C + 2D)(4) \\
1 &= A(1)(4) + B(4) + 2D(1)
\end{aligned}$$

B is known and C can be replaced by $-A$, so these become

$$\begin{aligned}
6A + 8D &= 0 \\
4A + 2D &= \frac{1}{5}
\end{aligned}$$

The solution is $A = \frac{2}{25},, B = \frac{1}{5},\ \ C = -\frac{2}{25},\ \ D = -\frac{3}{50}$.

So

$$\begin{aligned}
Y &= \frac{1}{2}\frac{2}{(s+1)^2 + 2^2} + \frac{2}{25}\frac{1}{(s+2)} + \frac{1}{5}\frac{1}{(s+2)^2} - \frac{2}{25}\frac{s+1}{(s+1)^2 + 2^2} - \frac{3}{50}\frac{2}{(s+1)^2 + 2^2} \\
y &= \frac{1}{2}e^{-t}\sin 2t + \frac{2}{25}e^{-2t} + \frac{1}{5}te^{-2t} - \frac{2}{25}e^{-t}\cos 2t - \frac{3}{50}e^{-t}\sin 2t.
\end{aligned}$$

5. (a) (15 pts.) Find the first five non-zero terms of the Fourier *cosine* series for the function

$$f(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ x, & \frac{1}{2} < x < 1 \end{cases} .$$

Solution:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

We have $L = 1$.

$$\begin{aligned}
a_0 &= 2\int_0^1 f(x)dx = 2\left(\int_0^{1/2} 0dx + \int_{1/2}^1 xdx\right) \\
&= 2\left[\frac{1}{2}x^2\right|_{1/2}^1 = \left(1 - \frac{1}{4}\right) = \frac{3}{4} \\
a_n &= 2\int_{1/2}^1 x\cos(n\pi x)dx = \frac{2}{(n\pi)^2}[\cos n\pi x + n\pi x\sin n\pi x|_{1/2}^1 \\
&= \frac{2}{(n\pi)^2}\left[(\cos n\pi - n\pi\sin n\pi) - \left(\cos\frac{n\pi}{2} + \frac{n\pi}{2}\sin\frac{n\pi}{2}\right)\right] \\
a_1 &= \frac{2}{\pi^2}\left[\cos\pi - \cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2}\right] = \frac{2}{\pi^2}\left[-1 - \frac{\pi}{2}\right] \\
a_2 &= \frac{2}{4\pi^2}\left[\cos 2\pi - \cos\pi - \frac{2\pi}{2}\sin\pi\right] = \frac{4}{4\pi^2} \\
a_3 &= \frac{2}{9\pi^2}\left[\cos 3\pi - \cos\frac{3\pi}{2} - \frac{3\pi}{2}\sin\frac{3\pi}{2}\right] = \frac{2}{9\pi^2}\left[-1 + \frac{3\pi}{2}\right] \\
a_4 &= \frac{2}{16\pi^2}\left[\cos 4\pi - \cos 2\pi - \frac{4\pi}{2}\sin 2\pi\right] = 0 \\
a_5 &= \frac{2}{25\pi^2}\left[\cos 5\pi - \cos\frac{5\pi}{2} - \frac{5\pi}{2}\sin\frac{5\pi}{2}\right] = \frac{2}{25\pi^2}\left[-1 - \frac{5\pi}{2}\right]
\end{aligned}$$

Finally, we can present, the requested terms of the series.

$$\begin{aligned}
f(x) \sim \frac{3}{8} &+ \left(\frac{2}{\pi^2}\left[-1 - \frac{\pi}{2}\right]\right)\cos(\pi x) + \left(\frac{1}{\pi^2}\right)\cos(2\pi x) + \left(\frac{2}{9\pi^2}\left[-1 + \frac{3\pi}{2}\right]\right)\cos(3\pi x) \\
&+ \left(\frac{2}{25\pi^2}\left[-1 - \frac{5\pi}{2}\right]\right)\cos(5\pi x) + \cdots
\end{aligned}$$

5(b) (10 pts.) To what value does the Fourier series of 5a converge at each of the following points?

(i) $x = -\frac{3}{4}$ $\frac{3}{4}$ (ii) $x = 0$ 0 (iii) $x = \frac{1}{2}$ $\frac{1}{4}$ (iv) $x = 1$ 1 (v) $x = \frac{3}{2}$ $\frac{1}{4}$

6 (25 pts.) Consider the following initial-boundary value problem for the heat equation.

$$\begin{array}{lll} \text{PDE} & u_t = 3u_{xx}, & 0 < x < \pi, \quad t > 0 \qquad (1) \\ \text{BC1} & u(0,t) = 0, & t > 0 \\ \text{BC2} & u_x(\pi,t) = 0, & t > 0 \qquad \# \\ \text{IC} & u(x,0) = 3\sin\frac{x}{2} - \sin\frac{11x}{2} + 7\sin\frac{19x}{2}, & 0 < x < \pi \end{array}$$

6 (a) Let the solution be $u(x,t) = X(x)T(t)$.. Use the method of separation of variables and the boundary conditions to obtain an eigenvalue problem for $X(x)$ and a differential equation for $T(t)$.

Separation of Variables:

$$u(x,t) = X(x)T(t)$$
$$X(x)T'(t) = 3X''(x)T(t)$$
$$\frac{X''}{X} = \frac{T'}{3T} = -\lambda$$

Two ordinary differential equations result.

$$X'' + \lambda X = 0$$
$$T' + 3\lambda T = 0$$

The boundary conditions lead to boundary conditions on X.

$$u(0,t) = X(0)T(t) = 0 \quad \Rightarrow X(0) = 0$$
$$u_x(\pi,t) = X'(\pi)T(t) = 0 \quad \Rightarrow X'(\pi) = 0$$

6 (b) Solve the eigenvalue problem.

Solution: The differential equation and boundary conditions for $X(x)$ in part (a) are the eigenvalue problem. The characteristic equation gives $r = \pm\sqrt{-\lambda}$. We look at the discriminant being positve, zero or negative.

Case 1. -$\lambda > 0$ $\quad -\lambda = \mu^2$.

$$X = c_1 e^{\upsilon x} + c_2 e^{-\mu x}$$
$$X' = \mu(c_1 e^{\upsilon x} - c_2 e^{-\mu x})$$
$$X(0) = (c_1 + c_2) = 0$$
$$c_1 = -c_2$$
$$X'(\pi) = \mu c_1(e^{\pi\mu} + e^{-\pi\mu}) = 0$$
$$c_1 = c_2 = 0$$

Case 2 $\quad -\lambda = 0$

$$\begin{aligned} X &= c_1 + c_2 x \\ X' &= c_2 \\ X(0) &= c_1 = 0 \\ X'(\pi) &= c_2 = 0 \end{aligned}$$

So $\lambda = 0$ is not an eigenvalue.

Case 3 $-\lambda < 0 \quad -\lambda = -\mu^2$.

$$\begin{aligned} X &= c_1 \cos \mu x + c_2 \sin \mu x \\ X' &= \mu(-c_1 \sin \mu x + c_2 \cos \mu x) \\ X(0) &= c_1 = 0 \\ X'(\pi) &= c_2 \cos(\pi\mu) = 0 \end{aligned}$$

So, non-zero solutions require $\cos(\pi\mu) = 0$. We have

$$\begin{aligned} \pi\mu &= (2n+1)\frac{\pi}{2} && n = 0, 1, 2, 3, \ldots \\ \mu_n &= \frac{2n+1}{2} && n = 1, 2, 3, \ldots \\ \lambda_n &= \left(\frac{2n+1}{2}\right)^2 && n = 1, 2, 3, \ldots \\ X_n &= c_n \sin\left(\frac{2n+1}{2}x\right) && n = 1, 2, 3, \ldots \end{aligned}$$

6 (c) For each eigenvalue, solve the corresponding differential equation for $T(t)$. For each eigenvalue give the corresponding solution to the heat equation.

Solution: The d.e. for T.

$$\begin{aligned} T' + 3\lambda T &= 0 \\ T' + 3\left(\frac{2n+1}{2}\right)^2 T &= 0 \end{aligned}$$

$$T_n = A_n \exp\left(-3\left(\frac{2n+1}{2}\right)^2 t\right)$$

6 (d) Give the formal series solution to the initial-boundary value problem. Apply the initial condition to determine the coefficients of the series. Give the final solution to the problem.

Solution:We combine the results.

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= A_n c_n \exp\left(-3\left(\frac{2n+1}{2}\right)^2 t\right)\sin\left(\frac{2n+1}{2}x\right) \end{aligned}$$

A formal solution is obtained by summing. (The two constants are combined in this step.)

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\ &= \sum_{n=0}^{\infty} b_n \exp\left(-3\left(\frac{2n+1}{2}\right)^2 t\right)\sin\left(\frac{2n+1}{2}x\right) \end{aligned}$$

To find the coefficients, we use the initial condition.

$$\begin{aligned} u(x,0) &= \sum_{n=0}^{\infty} b_n \sin\left(\frac{2n+1}{2}x\right) \\ &= 3\sin\frac{x}{2} - \sin\frac{11x}{2} + 7\sin\frac{19x}{2} \end{aligned}$$

Matching terms leads to $b_0 = 3$, $b_5 = -1$ and $b_9 = 7..$ All the rest are zero. With this, the solution is

$$u(x,t) = 3\exp\left(-3\left(\frac{1}{1}\right)^2 t\right)\sin\frac{x}{2} - \exp\left(-3\left(\frac{11}{2}\right)^2 t\right)\sin\frac{11x}{2} + 7\exp\left(-3\left(\frac{19}{2}\right)^2 t\right)\sin\frac{19x}{2}$$

7 (a) (15 pts.) Find the power series solution to the initial value problem

$$y'' + 2xy' - 3y = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

Be sure to give the recurrence relation for the coefficients of the power series. Give the first five nonzero terms of the solution.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

We substitute into the differential equation.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2\sum_{n=1}^{\infty} n a_n x^n - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

To combine the series, we adjust the indices. In the first, let $k = n - 2$. For the others, replace n by k.

$$\sum_{k=0}^{\infty} \{k+2\}(k+1) a_{k+2} x^k + 2\sum_{k=1}^{\infty} k a_k x^k - 3\sum_{k=0}^{\infty} a_k x^k = 0$$

The middle series does not have a constant term. We split that out and combine the rest.

$$(2a_2 - 3a_0) + \sum_{k=1}^{\infty} [\{k+2\}(k+1) a_{k+2} + (2k-3)s_k] x^k = 0$$

From the initial conditions,

$$a_0 = y(0) - 1. \quad a_1 = y'(0) = 0.$$

Then

$$2a_2 - 3a_0 = 0 \quad \Rightarrow a_2 = \frac{3}{2} a_0 = \frac{3}{2}$$

$$\{k+2\}(k+1) a_{k+2} + (2k-3)s_k = 0, \quad k = 1, 2, 3, \cdots$$

This can be written as the recurrence relation.

$$a_{k+2} = \frac{3 - 2k}{\{k+2\}(k+1)} a_k, \quad k = 1, 2, 3, \cdots$$

We observe that each coefficient comes from that which is two earlier, so all the odd terms are zero. We need three more non-zero terms.

$$a_4 = \frac{3-4}{4 \circ 3} a_2 = \frac{-3}{4!}$$

$$a_6 = \frac{-5}{6 \circ 5} a_4 = \frac{15}{6!}$$

$$a_8 = \frac{-9}{8 \circ 7} a_6 = \frac{-135}{8!}$$

The series solution is

$$y = 1 + \frac{3}{2}x^2 - \frac{3}{4!}x^4 + \frac{15}{6!}x^6 - \frac{135}{8!}x^8 + \cdots$$

7 (b) (15 pts.) Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0, \qquad 0 < x < 5$$
$$y'(0) = 0$$
$$y(5) = 0$$

Be sure to consider the cases $\lambda < 0, \lambda = 0,$ and $\lambda > 0.$
Solution: The characteristic equation is $r^2 + \lambda = 0$. Thus $r = \pm\sqrt{-\lambda}$. We consider the three cases of the quantity under the radical being positive, zero or negative.
Case 1. $-\lambda > 0$. We write $-\lambda = \mu^2$. The solution to the d.e is

$$y = c_1 e^{\mu x} + c_2 e^{-\mu x}$$
$$y' = \mu(c_1 e^{x} - c_2 e^{-\mu x})$$

From the boundary conditions,1

$$y'(0) = \mu(c_1 - c_2) = 0$$
$$c_1 = c_2$$
$$y(5) = c_1\left(e^{5\mu} + \frac{1}{e^{5\mu}}\right) = 0$$
$$c_1 = c_2 = 0$$

There is no non-zero solution in this case.
Case 2 $\quad -\lambda = 0 \quad$ The solution to the d.e is

$$y = c_1 + c_2 x$$
$$y' = c_2$$

From the boundary conditions,

$$y'(0) = c_2 = 0$$
$$y(5) = c_1 = 0$$

Again, there is no non-zero solution.

Case 3. $-\lambda < 0$ We write $-\lambda - -\mu^2$. $r = \pm\sqrt{-\mu^2} = \pm\mu i$. The solution to the d.e. is

$$y = c_1 \cos\mu x + c_2 \sin\mu x$$
$$y' = \mu(-c_1 \sin\mu x + c_2 \cos\mu x)$$

From the boundary conditions,

$$y'(0) = \mu c_2 = 0$$
$$c_2 = 0$$
$$y(5) = c_1 \cos 5\mu$$

For a non-zero solution, we must have

$$\cos 5\mu = 0$$
$$5\mu_n = (2n+1)\frac{\pi}{2} \qquad n = 0,1,2,..$$
$$\mu_n = (2n+1)\frac{\pi}{10} \qquad n = 0,1,2,\ldots$$

So the eigenvalues (λ_n) and corresponding eigenfunctions (y_n) are

$$\lambda_n = \mu_n^2 = \left[(2n+1)\frac{\pi}{10}\right]^2 \qquad n = 0,1,2,\ldots$$
$$y_n = c_n \cos\left(\frac{2n+1}{10}\pi x\right) \qquad n = 0,1,2,\ldots$$

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s) = \hat{f}(s)$		
1	$\frac{1}{s}$		$s > 0$
t^n	$\dfrac{n!}{s^{n+1}}$	$n \geq 1$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$		$s > a$
$\sin bt$	$\dfrac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$		$s > 0$
$e^{at}t^n$	$\dfrac{n!}{(s-a)^{n+1}}$	$n \geq 1$	$s > a$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$		$s > a$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$		$s > a$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		

Properties of Laplace Transforms

$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{f\}$
$\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$
$\mathcal{L}\left\{f'\right\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
$\mathcal{L}\left\{f''\right\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int x \cos bx dx = \frac{1}{b^2}(\cos bx + bx \sin bx) + C$
$\int x \sin bx dx = \frac{1}{b^2}(\sin bx - bx \cos bx) + C$
$\int \sec x dx = \ln\lvert\sec x + \tan x\rvert + C$
$\int \ln t dt = t \ln t - t + C$
$\int t \ln t dt = \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2 + C$
$\int t^2 \ln t dt = \frac{1}{3}t^3 \ln t - \frac{1}{9}t^3 + C$
$\int t \ln^2 t dt = \frac{1}{4}t^2(2\ln^2 t - 2\ln t + 1) + C$
$\int \frac{\ln t}{t} dt = \frac{1}{2}\ln^2 t + C$