Name:___________________________________ Lecturer __________________________

Lecture Section: ____________

Ma 221 14F Exam IA Solutions

Solve the following differential equations. Characterize your solution as explicit or implicit.

1 [25 **pts.**]

$$\frac{dy}{dx} = \frac{-y}{x} + \frac{5}{y^3}$$

Solution: This is a Bernoulli equation. We rewrite it as

$$\frac{dy}{dx} + \frac{y}{x} = 5y^{-3}$$

$$y^3\frac{dy}{dx} + \frac{1}{x}y^4 = 5$$

Let $v = y^4$. Then $v' = 4y^3y'$ and we may write the above DE as

$$\frac{1}{4}v' + \frac{1}{x}v = 5$$

$$v' + \frac{4}{x}v = 20$$

This is first order linear in v. The integrating factor is

$$e^{\int Pdx} = e^{\int \frac{4}{x}dx} = e^{4\ln x} = e^{\ln x^4} = x^4$$

Multiplying the DE by x^4 we have

$$x^4v' + 4x^3v = 20x^4$$

or

$$\frac{d\left(x^4v\right)}{dx} = 20x^4$$

Integrating we have

$$x^4v = 4x^5 + c$$

Thus

$$v = y^4 = 4x + \frac{c}{x^4}$$

This is an implicit solution.

Name:_________________________________ Lecturer _______________________

Lecture Section: ___________

2 $\left[25 \textbf{ pts}.\right]$

$$\frac{dy}{dx} = \frac{-\left[2x\cos\left(x^2+y^2\right) - 2\sin(2x)\right]}{[2y\cos\left(x^2+y^2\right) - 3\sin(3y)]}$$

Solution: First, we must do some algebra to put the d.e. into a standard form. From the y^2 term, we see that it is not linear and it's not separable. Let's try exact. Rewrite as

$$[2y\cos\left(x^2+y^2\right) - 3\sin(3y)]dy = -\left[2x\cos\left(x^2+y^2\right) - 2\sin(2x)\right]dx$$

$$\left[2x\cos\left(x^2+y^2\right) - 2\sin(2x)\right]dx + [2y\cos\left(x^2+y^2\right) - 3\sin(3y)]dy = 0$$

If we let $M = 2x\cos\left(x^2+y^2\right) - 2\sin(2x)$ and $N = 2y\cos\left(x^2+y^2\right) - 3\sin(3y)$, then

$$M_y = (-2x)(2y)\sin\left(x^2+y^2\right) = N_x$$

Hence this equation is exact. Thus there exists $f(x,y)$ such that

$$f_x = M = 2x\cos\left(x^2+y^2\right) - 2\sin(2x) \text{ and } f_y = N = 2y\cos\left(x^2+y^2\right) - 3\sin(3y)$$

Integrating f_x with respect to x leads to

$$f = \sin\left(x^2+y^2\right) + \cos(2x) + g(y)$$

Therefore

$$f_y = 2y\cos\left(x^2+y^2\right) + g'(y) = N = 2y\cos\left(x^2+y^2\right) - 3\sin(3y)$$

We see that

$$g'(y) = -3\sin(3y)$$

so a choice for $g(y)$ is

$$g(y) = \cos(3y)$$

Hence

$$f = \sin\left(x^2+y^2\right) + \cos(2x) + \cos(3y)$$

and the implicit solution is given by

$$f = \sin\left(x^2+y^2\right) + \cos(2x) + \cos(3y) = C$$

Name:_________________________________ Lecturer ________________________

Lecture Section: ___________

3 [25 **points**]

$$\frac{dy}{dx} = \frac{2y}{x} + x^2 \cos x \qquad y(\pi) = 2$$

Solution:We may rewrite the equation as

$$\frac{dy}{dx} - \frac{2y}{x} = x^2 \cos x$$

This is a first order linear DE. The integrating factor is

$$e^{\int P(t)dx} = e^{\int \frac{-2}{x}dx} = e^{-2\ln|x|} = e^{\ln(x^{-2})} = x^{-2}$$

Multiplying the DE by x^{-2} we have

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = \cos x$$

or

$$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = \cos x$$

$$\frac{1}{x^2}y = \sin x + c$$

The explicit solution of the d.e. is

$$y = x^2(c + \sin x)$$

The initial condition $y(\pi) = 2$ implies

$$2 = \pi^2(c + \sin \pi)$$

so

$$c = \frac{2}{\pi^2}$$

and the explicit solution is given by

$$y = x^2\left(\frac{2}{\pi^2} + \sin x\right)$$

Name:_________________________________ Lecturer ________________________

Lecture Section: ___________

4 [25 **pts**.]

$$\frac{dy}{dx} = \frac{\cos^2 x}{\sin^2 y} \qquad y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$$

Solution: This equation is separable, since it may be rewritten as

$$\sin^2 y\, dy = \cos^2 x\, dx$$

Integrating, using the integral table, we have

$$\frac{1}{2}y - \frac{1}{4}\pi - \frac{1}{4}\sin 2y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

The initial condition implies

$$\frac{\pi}{8} - \frac{\pi}{4} - \frac{1}{4}\sin\frac{\pi}{2} = \frac{\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2} + C$$

$$-\frac{\pi}{4} - \frac{1}{4} = \frac{1}{4} + C$$

$$C = -\frac{\pi}{4} - \frac{1}{2}$$

and the implicit solution is solution is

$$\frac{1}{2}y - \frac{1}{4}\sin 2y = \frac{1}{2}x + \frac{1}{4}\sin 2x - \frac{1}{2}$$

Name:___________________________________ Lecturer __________________________

Lecture Section: ___________

Table of Integrals

$\int \sec^2 t dt = \tan t + C$
$\int \tan t dt = \ln(\sec t) + C$
$\int \frac{\sec^2 t}{\tan t} dt = \ln(\tan t) + C$
$\int t e^{at} dt = \frac{1}{a^2} e^{at}(at - 1) + C$
$\int t^2 e^{at} dt = \frac{1}{a^3} e^{at}\left(a^2 t^2 - 2at + 2\right) + C$
$\int \cos^2 t dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + C$
$\int \cos^3 t dt = \frac{1}{3} \cos^2 t \sin t + \frac{2}{3} \sin t + C$
$\int \sin^2 t dt = \frac{1}{2} t - \frac{1}{4} \pi - \frac{1}{4} \sin 2t + C$
$\int \sin^3 t dt = \frac{1}{12} \cos 3t - \frac{3}{4} \cos t + C$