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Lecturer _____

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**Ma 221
14F****Exam IB Solutions**

Solve the following differential equations. Characterize your solution as explicit or implicit.

1 [25 pts.]

$$\frac{dy}{dx} = \frac{-y}{x} + \frac{4}{y^2}$$

Solution: This is a Bernoulli equation. We rewrite it as

$$\frac{dy}{dx} + \frac{y}{x} = 4y^{-2}$$

$$y^2 \frac{dy}{dx} + \frac{1}{x} y^3 = 4$$

Let $v = y^3$. Then $v' = 3y^2 y'$ and we may write the above DE as

$$\frac{1}{3} v' + \frac{1}{x} v = 4$$

$$v' + \frac{3}{x} v = 12$$

This is first order linear in v . The integrating factor is

$$e^{\int P dx} = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = e^{\ln x^3} = x^3$$

Multiplying the DE by x^3 we have

$$x^3 v' + 3x^2 v = 12x^3$$

or

$$\frac{d(x^3 v)}{dx} = 12x^3$$

Integrating we have

$$x^3 v = 3x^4 + c$$

Thus

$$v = y^3 = 3x + \frac{c}{x^3}$$

This is an implicit solution.

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2 [25 pts.]

$$\frac{dy}{dx} = \frac{-[2x \sin(x^2 + y^2) + 3 \cos(3x)]}{[2y \sin(x^2 + y^2) + 2 \cos(2y)]}$$

Solution: First, we must do some algebra to put the d.e. into a standard form. From the y^2 term, we see that it is not linear and it's not separable. Let's try exact. Rewrite as

$$[2y \sin(x^2 + y^2) + 2 \cos(2y)]dy = -[2x \sin(x^2 + y^2) + 3 \cos(3x)]dx$$

$$[2x \sin(x^2 + y^2) + 3 \cos(3x)]dx + [2y \sin(x^2 + y^2) + 2 \cos(2y)]dy = 0$$

If we let $M = 2x \sin(x^2 + y^2) + 3 \cos(3x)$ and $N = 2y \sin(x^2 + y^2) + 2 \cos(2y)$, then

$$M_y = (2x)(2y) \cos(x^2 + y^2) = N_x$$

Hence this equation is exact. Thus there exists $f(x, y)$ such that

$$f_x = M = 2x \sin(x^2 + y^2) + 3 \cos(3x) \text{ and } f_y = N = 2y \sin(x^2 + y^2) + 2 \cos(2y)$$

Integrating f_x with respect to x leads to

$$f = -\cos(x^2 + y^2) + \sin(3x) + g(y)$$

Therefore

$$f_y = 2y \sin(x^2 + y^2) + g'(y) = N = 2y \sin(x^2 + y^2) + 2 \cos(2y)$$

We see that

$$g'(y) = 2 \cos(2y)$$

so a choice for $g(y)$ is

$$g(y) = \sin(2y)$$

Hence

$$f = -\cos(x^2 + y^2) + \sin(3x) + \sin(2y)$$

and the implicit solution is given by

$$f = -\cos(x^2 + y^2) + \sin(3x) + \sin(2y) = C$$

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3 [25 points]

$$\frac{dy}{dx} = \frac{3y}{x} + x^3 \sin x \quad y\left(\frac{\pi}{2}\right) = 8$$

Solution: We may rewrite the equation as

$$\frac{dy}{dx} - \frac{3y}{x} = x^3 \sin x$$

This is a first order linear DE. The integrating factor is

$$e^{\int P(x)dx} = e^{\int -\frac{3}{x}dx} = e^{-3\ln|x|} = e^{\ln(x^{-3})} = x^{-3}$$

Multiplying the DE by x^{-3} we have

$$\frac{1}{x^3} \frac{dy}{dx} - \frac{3}{x^4} y = \sin x$$

or

$$\frac{d}{dx} \left(\frac{1}{x^3} y \right) = \sin x$$

$$\frac{1}{x^3} y = -\cos x + c$$

The explicit solution of the d.e. is

$$y = x^3(c - \cos x)$$

The initial condition $y(\pi/2) = 8$ implies

$$8 = \left(\frac{\pi}{2}\right)^3 \left(c - \cos \frac{\pi}{2}\right)$$

so

$$c = 8\left(\frac{2}{\pi}\right)^3 = \left(\frac{4}{\pi}\right)^3$$

and the explicit solution is given by

$$y = x^3 \left(\left(\frac{4}{\pi}\right)^3 - \cos x \right)$$

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4 [25 pts.]

$$\frac{dy}{dx} = \frac{\sin^2 x}{\cos^2 y} \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$$

Solution: This equation is separable, since it may be rewritten as

$$\cos^2 y \, dy = \sin^2 x \, dx$$

Integrating, using the integral table, we have

$$\frac{1}{2}y + \frac{1}{4} \sin 2y = \frac{1}{2}x - \frac{\pi}{4} - \frac{1}{4} \sin 2x + C$$

The initial condition implies

$$\frac{\pi}{8} + \frac{1}{4} \sin \frac{\pi}{2} = \frac{\pi}{8} - \frac{\pi}{4} - \frac{1}{4} \sin \frac{\pi}{2} + C$$

$$\frac{1}{4} = -\frac{\pi}{4} - \frac{1}{4} + C$$

$$C = +\frac{\pi}{4} + \frac{1}{2}$$

and the implicit solution is

$$\frac{1}{2}y + \frac{1}{4} \sin 2y = \frac{1}{2}x - \frac{1}{4} \sin 2x + \frac{1}{2}$$

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Table of Integrals

$\int \sec^2 t dt = \tan t + C$
$\int \tan t dt = \ln(\sec t) + C$
$\int \frac{\sec^2 t}{\tan t} dt = \ln(\tan t) + C$
$\int t e^{at} dt = \frac{1}{a^2} e^{at} (at - 1) + C$
$\int t^2 e^{at} dt = \frac{1}{a^3} e^{at} (a^2 t^2 - 2at + 2) + C$
$\int \cos^2 t dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + C$
$\int \cos^3 t dt = \frac{1}{3} \cos^2 t \sin t + \frac{2}{3} \sin t + C$
$\int \sin^2 t dt = \frac{1}{2} t - \frac{1}{4} \pi - \frac{1}{4} \sin 2t + C$
$\int \sin^3 t dt = \frac{1}{12} \cos 3t - \frac{3}{4} \cos t + C$