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Ma 221 **Exam II B Solutions** **14F**

1. (30 **pts**. **total**) Consider the differential equation

$$L[y] = y'' + 2y' - 8y = 16e^{-2x} - 32x^2$$

1 a (6 **pts**.) Find the solution of the corresponding homogeneous equation $L[y] = 0$.

Solution: The characteristic equation is

$$p(r) = r^2 + 2r - 8 = (r+4)(r-2) = 0.$$

Thus $r = 2, -4$ and

$$y_h = c_1 e^{2x} + c_2 e^{-4x}.$$

1 b (20 **pts**.) Find a particular solution of this equation.

Solution: We first find y_{p_1} for $16e^{-2x}$. Since $p(-2) = 4 - 4 - 8 = -8,$ we have

$$y_{p_1} = \frac{16e^{-2x}}{-8} = -2e^{-2x}.$$

To find y_{p_2} corresponding to -$32x^2$ we let

$$y_{p_2} = Ax^2 + Bx + C$$

Then

$$y'_{p_2} = 2Ax + B$$
$$y''_{p_2} = 2A$$

Plugging into the DE we have

$$2A + 4Ax + +2B - 8Ax^2 - 8Bx - 8C = -32x^2$$

$$-8A = -32$$
$$4A - 8B = 0$$
$$2A + 2B - 8C = 0$$

, Solution is: $\left[A = 4, B = 2, C = \frac{3}{2}\right]$

Thus

$$y_{p_2} = 4x^2 + 2x + \frac{3}{2}$$
$$y_p = y_{p_1} + y_{p_2}$$
$$= -2e^{-2x} + 4x^2 + 2x + \frac{3}{2}$$

1 c (4 **pts**.) Give a general solution of the equation

$$y'' + 2y' - 8y = 16e^{-2x} - 32x^2$$

Solution:

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^{2x} + c_2 e^{-4x} - 2e^{-2x} + 4x^2 + 2x + \frac{3}{2}$$

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2 (25 **pts**) Find a particular solution of the differential equation

$$y'' - 4y = 4t\sin 2t$$

Solution:
We look for a solution in the form

$$y_p(t) = (At+B)\cos 2t + (Ct+D)\sin 2t$$

$$y_p'(t) = A\cos 2t - 2(At+B)\sin 2t + C\sin 2t + 2(Ct+D)\sin 2t$$
$$y_p''(t) = -2A\sin 2t + 2A\sin 2t - 4(At+B)\cos 2t + 2C\cos 2t + 2C\cos 2t - 4(Ct+D)\sin 2t$$

Subtuting into the DE yields

$$\begin{aligned} y_p''(t) - 4y_p(t) &= (-4At - 4B + 4C - 4At - 4B)\cos 2t + (-4A - 4Ct - 4D - 4Ct - 4D)\sin 2t \\ &= 4t\sin 2t \end{aligned}$$

Hence, equating terms yields

$$\begin{aligned} t\cos 2t :&\quad -4A - 4A = 0 \\ t\sin 2t :&\quad -4C - 4C = 4 \\ \cos 2t :&\quad -8B + 4C = 0 \\ \sin 2t :&\quad -4A - 8D = 0 \end{aligned}$$

So $A = D = 0$, $B = -\frac{1}{4}$, $C = -\frac{1}{2}$ and the solution is

$$y_p = -\frac{1}{2}t\sin 2t - \frac{1}{4}\cos 2t.$$

3 (25 **pts**.) Solve the equation

$$y'' + 4y = \sec 2x$$

Solution: We first find the homogeneous solution. The characteristic equation is

$$r^2 + 4 = 0$$

Thus $r^2 = -4$, $r = \pm i$ so

$$y_h = c_1\cos 2x + c_2\sin 2x$$

To find a particular solution we use the Method of Variation of Parameters and let $y_1 = \cos 2x$ and $y_2 = \sin 2x$

$$y_p = v_1\cos 2x + v_2\sin 2x$$

Then the two equations for v_1' and v_2' ,

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1 + v_2' y_2 = \frac{f}{a}$$

become

$$v_1' \cos 2x + v_2' \sin 2x = 0$$
$$v_1'(-2\sin 2x) + v_2'(2\cos 2x) = \sec 2x$$

or

$$\cos 2x v_1' + \sin 2x v_2' = 0$$
$$-\sin 2x v_1' + \cos 2x v_2' = \frac{1}{2}\frac{1}{\cos 2x}$$

Multiply the first equation by $\sin 2x$, the second by $\cos 2x$ and add to obtain

$$\left(\sin^2 2x + \cos^2 2x\right) v_2' = v_2' = \frac{1}{2}$$
$$v_2 = \frac{1}{2}x + c_2$$

Then from the first equation we have

$$v_1' = -\frac{1}{2}\frac{\sin 2x}{\cos 2x}$$

From the table of integtrals below

$$v_1 = \frac{1}{4}[\ln(\cos 2x)] + c_1$$

so

$$y = v_1 \cos 2x + v_2 \sin 2x = \left(\frac{1}{4}[\ln(\cos 2x)] + c_1\right)\cos 2x + \left(\frac{1}{2}x + c_2\right)\sin 2x.$$

4 (20 **pts**.) Solve

$$x^2 y'' - xy' - 3y = 0$$

Solution: This is a Cauchy-Euler equation. The indicial (auxiliary) equation for a solution of the form x^r is

$$r(r-1) - r - 3 = r^2 - 2r - 3 = 0$$
$$(r+1)(r-3) = 0$$

So the roots are $r = -1$ and $r = 3$. Thus, the solution is

$$y = c_1 x^{-1} + c_2 x^3.$$

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Table of Integrals

$$\int \ln t\,dt = t(\ln t - 1) + C$$

$$\int t \ln t = \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2 + C$$

$$\int (\ln t)^2 dt = t\left(\ln^2 t - 2\ln t + 2\right) + C$$

$$\int \frac{\ln t}{t} dt = \frac{1}{2}\ln^2 t + C$$

$$\int \sec u\,du = \ln(\sec u + \tan u) + C$$

$$\int \tan u\,du = \int \frac{\sin u}{\cos u} du = -\ln(\cos u) + C$$

$$\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$$

$$\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$$

$$\int u \sin u\,du = \sin u - u\cos u + C$$

$$\int u \cos u\,du = \cos u + u\sin u + C$$

An Identity

$$\sin u \tan u = \frac{\sin^2 u}{\cos u} = \frac{1 - \cos^2 u}{\cos u} = \sec u - \cos u$$