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Ma 221 Exam IIIB Solutions 14F

1a (10 **pts**.)

$$f(t) = \left\{ \begin{cases} 0 & if \quad 0 \le t < 3 \\ 5 & if \quad 3 \le t \le 7 \\ 0 & if \quad 7 < t \end{cases} \right.$$

Use the definition of the Laplace transform to determine the Laplace transform of $f(t)$.

Solution:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)dt = \int_3^7 5e^{-st}dt \\ &= 5\frac{e^{-st}}{-s}\Big|_3^7 = \frac{5(e^{-7s} - e^{-3s})}{-s} \\ &= \frac{5(e^{-7s} - e^{-5s})}{s} \end{aligned}$$

1b (15 **pts**.) Determine

$$\mathcal{L}^{-1}\left\{\frac{5s+7}{s^2+4s+13}\right\}$$

Solution:

$$\begin{aligned} \frac{5s+7}{s^2+4s+13} &= \frac{5s+7}{(s+2)^2+9} = \frac{5s+7}{(s+2)^2+3^2} \\ &= \frac{5(s+2)}{(s+2)^2+3^2} + \frac{-3}{(s+2)^2+3^2} \\ &= \frac{5(s+2)}{(s+2)^2+3^2} - \frac{3}{(s+2)^2+3^2} \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{5s+7}{s^2+4s+13}\right\} &= 5\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+3^2}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\} \\ &= 5e^{-2t}\cos 3t - e^{-2t}\sin 3t \end{aligned}$$

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2a $\left(15\ \textbf{pts.}\right)$ Consider the initial value problem

$$y'' + 2y' + y = 18e^{2t} \quad y(0) = 6 \;\; y'(0) = -4$$

Let $Y(s) = \mathcal{L}\{y\}(s)$. Use Laplace transforms to show that

$$Y(s) = \frac{18}{(s-2)(s+1)^2} + \frac{6s}{(s+1)^2} + \frac{8}{(s+1)^2}$$

Solution: Taking the Laplace transform of both sides of the DE we have

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 18\mathcal{L}\{e^{2t}\}$$

or letting $Y(s) = \mathcal{L}\{y\}(s)$

$$s^2Y(s) - sy(0) - y'(0) + 2\{Y(s) - y(0)\} + Y(s) = \frac{18}{s-2}$$

Using the given initial conditions we have

$$(s^2 - 2s + 1)Y(s) - 6s - 8 = \frac{18}{s+2}$$

Thus

$$Y(s) = \frac{18}{(s+2)(s-1)^2} + \frac{6s}{(s-1)^2} + \frac{8}{(s-1)^2}$$

2b $\left(15\ \textbf{pts.}\right)$ Find the solution to the initial problem above, namely,

$$y'' + 2y' + y = 18e^{2t} \quad y(0) = 6 \;\; y'(0) = -4$$

by finding

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{18}{(s-2)(s+1)^2} + \frac{6s}{(s+1)^2} + \frac{8}{(s+1)^2}\right\}$$

Solution: The last two fractions can be easily dealt with by a little algebra.

$$\frac{6s+8}{(s+1)^2} = \frac{6(s+1)+2}{(s+1)^2} = \frac{6}{s+1} + \frac{2}{(s+1)^2}$$

In order to invert $Y(s)$ we need to do a partial fractions breakup of

$$\frac{18}{(s-2)(s+1)^2}.$$

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We have

$$\frac{18}{(s-2)(s+1)^2} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Multipication by the common denominator gives

$$18 = A(s+1)^2 + B(s-2)(s+1) + C(s-2)$$

Set $s = -1$

$$18 = -3C$$

So $C = -6$. Set $s = 2$

$$18 = 9A$$

Thus $A = 2$. Equate the coefficient of s^2 on each side of the equation.

$$0 = A + B$$

$B = -A = -2$. Now, combining everything, we have

$$\begin{aligned} Y(s) &= \frac{18}{(s-2)(s+1)^2} + \frac{6s}{(s+1)^2} + \frac{8}{(s+1)^2} \\ &= \frac{2}{s-2} + \frac{-2}{s+1} + \frac{-6}{(s+1)^2} + \frac{6}{s+1} + \frac{2}{(s+1)^2} \\ &= \frac{2}{s-2} + \frac{4}{s+1} + \frac{-4}{(s+1)^2} \end{aligned}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s-2} + \frac{4}{s+1} + \frac{-4}{(s+1)^2}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \\ &= 2e^{2t} + 4e^{-t} - 4te^{-t} \end{aligned}$$

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3 (25 **pts**.) Find the first 5 nonzero terms of the power series solution about $x = 0$ for the DE:

$$y'' - 3xy' = 0$$

Be sure to give the recurrence relation.
Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

so

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

.
The differential equation ⇒

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - 3\sum_{n=1}^{\infty} a_n n x^n = 0$$

Shifting the first series by letting $n - 2 = k$ or $n = k + 2$ we have

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - 3\sum_{n=1}^{\infty} a_n n x^n = 0$$

Replacing n by k in the second series we have

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - 3\sum_{k=1}^{\infty} a_k k x^k = 0$$

Since the first series has one more term, we have

$$2a_2 + \sum_{k=1}^{\infty} [a_{k+2}(k+2)(k+1) - 3a_k k]x^k = 0$$

Thus

$$a_2 = 0$$

and we have the recurrence relation

$$a_{k+2}(k+2)(k+1) - 3a_k k = 0 \quad \text{for } k = 1, 2, 3, \ldots$$

or

$$a_{k+2} = \frac{3k}{(k+2)(k+1)} a_k \quad \text{for } k = 1, 2, 3, \ldots$$

Therefore

$$a_3 = \frac{3 \cdot 1}{3 \cdot 2} a_1$$

$$a_4 = \frac{3 \cdot 2}{4 \cdot 3} a_2 = 0$$

$$a_5 = \frac{3 \bullet 3}{5 \bullet 4} a_3 = \frac{(3)^2 \bullet 3}{5!} a_1$$
$$a_6 = \frac{3 \bullet 4}{6 \bullet 5} a_4 = 0$$
$$a_7 = \frac{3 \bullet 5}{7 \bullet 6} a_5 = \frac{(3)^3 \bullet 5 \bullet 3}{7!} a_1$$

Thus

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 \left[x + \frac{3}{3 \bullet 2} x^3 + \frac{(3)^2 \bullet 3}{5!} x^5 + \frac{(3)^3 \bullet 5 \bullet 3}{7!} x^7 + \ldots \right]$$

4 (25 **pts**.) Find all eigenvalues (λ) and the corresponding eigenfunctions for the boundary value problem

$$y'' - 3y + \lambda y = 0 \qquad y(0) = y'(2) = 0$$

Be sure to consider all values of λ.

Solution: This is an equation with constant coefficients. The characteristic equation is

$$r^2 - 3 + \lambda = 0$$

Thus

$$r = \sqrt{3 - \lambda}$$

There are 3 cases to consider: $3 - \lambda > 0,$ $3 - \lambda = 0,$ and $3 - \lambda < 0.$

Case I: $3 - \lambda > 0,$ that is $\lambda < 3.$ Let $3 - \lambda = \mu^2 \neq 0.$ We have the two roots $r = \pm\upsilon,$ and the solution

$$y(x) = c_1 e^{\mu s} + c_2 e^{-\mu x}$$

$$y(0) = c_1 + c_2 = 0$$

so $c_1 = -c_2$ and

$$y(x) = c_1 (e^{\mu s} - e^{-\mu x})$$

Then

$$y'(x) = \mu c_1 (e^{\mu s} + e^{-\mu x})$$
$$y'(2) = \mu c_1 (e^{2\mu} + e^{-2\mu}) = 0$$

Since $e^{2\mu} + e^{-2\mu} \neq 0$ this implies that $c_1 = 0$ and therefore $c_2 = 0$ and the only solution for $\lambda < 3$ is the trivial solution $y = 0.$ Hence there are no eigenvalues if $\lambda < 3.$

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Case II: $\lambda = 3.$ Then $r = 0$ is a repeated root, and

$$y(x) = c_1 + c_2 x$$

$$y(0) = c_1 = 0$$

$$y'(x) = c_2$$

$$y'(2) = c_2 = 0$$

Therefore $c_2 = 0$ and $\lambda = 3$ is not an eigenvalue.

Case III. $3 - \lambda < 0,$ that is $\lambda > 3.$ Let $3 - \lambda = -\mu^2 \neq 0.$ We have the complex roots $r = \pm\mu i$ and

$$y(x) = [c_1 \cos(\mu x) + c_2 \sin(\mu x)]$$

$$y(0) = c_1 = 0$$

Thus

$$y'(x) = \mu c_2 \cos(\mu x)$$

$$y'(2) = \mu c_2 \cos(2\mu)$$

So, for a non-zero solution, we must have

$$\cos(2\mu) = 0$$

Thus 2μ must be an odd multiple of $\frac{\pi}{2}$.

$$2\mu = (2n+1)\frac{\pi}{2} \qquad n = 0, 1, 2, 3, \ldots$$

or

$$\mu_n = (2n+1)\frac{\pi}{4} \qquad n = 0, 1, 2, 3, \ldots$$

And finally

$$\lambda_n = 3 + \mu^2 = 3 + \left((2n+1)\frac{\pi}{4}\right)^2 \quad n = 0, 1, 2, 3, \ldots$$

are the eigenvalues with corresponding eigenfunctions

$$y(x) = c_n \sin\left(\frac{2n+1}{4}\pi x\right).$$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f\}(s) = \hat{f}(s)$		
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$		$s > a$
$\sin bt$	$\dfrac{b}{s^2+b^2}$		$s > 0$
$\cos bt$	$\dfrac{s}{s^2+b^2}$		$s > 0$
$e^{at}f(t)$	$\mathscr{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}(\mathscr{L}\{f\}(s))$		