Ma 221 **Final Exam Solutions**
12/16/14

Print Name: ____ **Lecture Section**: _________

1.

(a) (8 pts) Solve

$$\frac{dy}{dx} = \frac{\sin x}{2e^y} \quad y(0) = 0.$$

Solution: It is a separable equation. Separate the variables

$$2e^{-y}dy = \sin x dx.$$

Integrate

$$-2e^{-y} = -\cos x + C.$$

Use the initial condition $y(0) = 0$

$$-2 = -1 + C \Rightarrow C = -1.$$

Thus, the implicit solution is

$$-2e^{-y} = -\cos x - 1$$

(b) (7 pts) Solve

$$(2x\cos y + 1)dx + \left(-x^2 \sin y + 2y\right)dy = 0.$$

Solution: The d.e. is not linear nor separable, check if it is exact. Let

$$M = 2x\cos y + 1, \quad N = -x^2 \sin y + 2y.$$

The derivatives

$$M_y = -2x\sin y, N_x = -2x\sin y$$

are equal and continuous and so it is exact. Using $F_x = M$

$$F = \int(2x\cos y + 1)dx = x^2\cos y + x + g(y).$$

Using $F_y = N$

$$-x^2 \sin y + g' = -x^2 \sin y + 2y \Rightarrow g'(y) = 2y.$$

Hence we take $g(y)$ to be an antiderivative of $-2y$

$$g(y) = y^2 \Rightarrow F(x,y) =.x^2\cos y + x + y^2$$

The implicit solution is

$$x^2\cos y + x + y^2 = c$$

1 (c) (10 pts) Find a general solution of

$$x^2y'' + 3xy' + 5y = 0.$$

Solution: This is a Caughy-Euler (or equi-dimensional) equation. We look for a solution of the form $y = x^r$. Substitution gives

$$r(r-1)x^r + 3rx^r + 5x^r = 0$$
$$\left(r^2 + 2r + 5\right)x^r = 0$$
$$r^2 + 2r + 1 = -4$$
$$(r+1)^2 = -4$$
$$r = -1 \pm 2i$$

One could also use the quadratic formula to solve the indicial equation. So the solutions come from the real and imaginary parts of one of the complex solutions.

$$\begin{aligned} x^r &= x^{-1+2i} = x^{-2}x^{2i} \\ &= x^{-1}\left(e^{\ln x}\right)^{2i} = x^{-1}e^{2i\ln x} \\ &= x^{-1}[\cos(2\ln x) + i\sin(2\ln x)] \end{aligned}$$

Finally, a general solution of the d.e. is

$$y = c_1 x^{-1}\cos(2\ln x) + c_2\sin(2\ln x).$$

2. (a) (12 pts) Find a general solution of

$$y'' - 2y' = 4x + 2 - 10\sin x.$$

Solution: First, the auxillary equation $r^2 - 2r = 0$ has the roots $r = 0, 2$ and the homogeneous equation has a general solution

$$y_h = C_1 + C_2 e^{2x}.$$

- Let us use the method of undetermined coefficients. For $f_1(x) = 4x + 2$, we see that $\alpha = 0$ is a single root of the auxillary equation, and therefore we take

  $$y_{p_1} = x(Ax + B).$$

  Substitute it in the equation with f_1

  $$2A - 2(2Ax + B) = 4x + 2,$$

  which leads to

  $$-4A = 4,\ 2A - 2B = 2 \Rightarrow A = -1,\ B = -2 \Rightarrow y_{p_1} = -x^2 - 2x.$$

- For $f_2(x) = -10\sin x$ there are two ways to find y_{p_2}.
  First approach: since $\beta = i$ is not a root of the auxillary equation we take

  $$y_{p_2} = A\cos x + B\sin x.$$

  Substitute it into the equation to get

  $$-A\cos x - B\sin x - 2(-A\sin x + B\cos x) = -10\sin x,$$

  $$(-A - 2B)\cos x + (-B + 2a)\sin x = -10\sin x,$$

  $$A = -4, \quad B = 2,$$

  $$y_{p_2} = -4\cos x + 2\sin x.$$

Second approach: add the equation with $f_2(x)$ to its comlimentary equation to get

$$w'' - 2w'^{ix}.$$

Since $\alpha = i$ is not a root of the auxillary eqaution

$$w_p = \frac{-10e^{ix}}{i^2 - 2i} = \frac{10}{1 + 2i}e^{ix}$$

$$= 2(1 - 2i)e^{ix} = 2(1 - 2i)(\cos x + i\sin x).$$

$$y_{p_2} = \operatorname{Im} w_p = -4\cos x + 2\sin x.$$

- Using the superposition principle

  $$y_p = -x^2 - 2x - 4\cos x + 2\sin x.$$

- Hence, a general solution is

$$\begin{aligned} y &= y_h + y_p \\ C_1 + C_2 e^{2x} &= -x^2 - 2x - 4\cos x + 2\sin x. \end{aligned}$$

2(b) (13 pts.) Find a general solution of

$$y'' - 2y' + y = e^x \ln x.$$

Solution: First, the auxillary equation $r^2 - 2r + 1 = 0$ has the double root 0 and the homogeneous equation has a general solution

$$y_h = C_1 e^x + C_2 x e^x.$$

- We can not use the method of undetermined coefficients to find a particular solution. Let us use the method of variation of parameters. Assume

  $$y_p = v_1(x) e^x + v_2(x) x e^x.$$

  Find $v_1(x)$ and $v_2(x)$ using the system of equations (we can use direct formulas as well):

  $$\begin{cases} v_1' e^x + v_2' x e^x = 0, \\ v_1' e^x + v_2'(e^x + x e^x) = e^x \ln x. \end{cases}$$

  Subtract the first equation from the second and divide by e^x to get $v_2' = \ln x$ and then from the first equation we obtain $v_1' = -x v_2' = -x \ln x$. Find antiderivatives

  $$v_1 = \int -x \ln x dx = -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2,$$

  $$v_2 = \int \ln x dx = x \ln x - x.$$

  Hence, a particular solution is

  $$\begin{aligned} y_p &= (-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2)e^x + (x \ln x - x) x e^x \\ &= \frac{1}{2}x^2 \ln x e^x - \frac{3}{4}x^2 e^x. \end{aligned}$$

  and a general solution is

  $$y = C_1 e^x + C_2 x e^x + \frac{1}{2}x^2 \ln x e^x - \frac{3}{4}x^2 e^x.$$

3. (a) (10 pts.) Let

$$g(t) = \begin{cases} t \ \text{ for } 0 < t < 1 \\ \\ e^t \ \ \text{ for } 1 < t < \infty \end{cases}.$$

Use the definition of the Laplace transform to find $\mathcal{L}\{g(t)\}$.
Solution: By definition

$$\mathcal{L}\{g\} = \int_0^\infty e^{-st} g(t)dt = \int_0^1 e^{-st} t dt + \int_1^\infty e^{-st} e^t dt.$$

For the first integral

$$\int_0^1 e^{-st} t dt = -\frac{1}{s} t e^{-st} - \frac{1}{s^2} e^{-st} \Big|_0^1 = -\frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-s} + \frac{1}{s^2}.$$

For the second integral

$$\int_1^\infty e^{-st} e^t dt = \lim_{N\to\infty} \int_1^N e^{-st} e^t dt = \lim_{N\to\infty} \frac{e^{(1-s)t}}{1-s} \Big|_1^N$$

$$= \lim_{N\to\infty} \frac{e^{(1-s)N} - e^{1-s}}{1-s} = \frac{e^{1-s}}{s-1} \text{ for } s > 1.$$

For $s < 1$ it diverges, and for $s = 1$ it also diverges as $\int_1^\infty dt$. Hence, for $s > 1$

$$\mathcal{L}\{g\} = \frac{-(s+1)}{s^2} e^{-s} + \frac{1}{s^2} + \frac{e^{1-s}}{s-1}.$$

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(b) (15 pts.) Solve using Laplace Transforms:

$$y'' - 2y' - 3y = 16e^{-t} \qquad y(0) = 1,\ y'(0) = 3.$$

Solution: Let $Y = \mathcal{L}\{y\}$. Apply the Laplace transform to the equation

$$s^2Y - s - 3 - 2(sY - 1) - 3Y = \frac{16}{s+1},$$

$$(s^2 - 2s - 3)Y = \frac{16}{s+1} + s + 1,$$

$$Y = \frac{s^2 + 2s + 17}{(s+1)^2(s-3)}.$$

Find the inverse Laplace transform of Y. We have

$$\frac{s^2 + 2s + 17}{(s+1)^2(s-3)} = \frac{A}{s-3} + \frac{B}{s+1} + \frac{C}{(s+1)^2},$$

$$s^2 + 2s + 17 = A(s+1)^2 + B(s-3)(s+1) + C(s-3).$$

Substitute $s = 3$ to get

$$9 + 6 + 17 = A4^2 \Rightarrow 32 = A16 \Rightarrow A = 2.$$

Substitute $s = -1$ to get

$$1 - 2 + 17 = -4C \Rightarrow 16 = -4C \Rightarrow C = -4.$$

Compare the coefficients at s^2

$$1 = A + B \Rightarrow B = -1.$$

Therefore

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{\frac{2}{s-3}\} - \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{4}{(s+1)^2}\}$$

$$= 2e^{3t} - e^{-t} - 4te^{-t}.$$

4.) a.) (10 pts.) Use separation of variables, $u(x,t) = X(x)T(t)$, to find two ordinary differential equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$e^{x-t}\frac{\partial^2 u}{\partial x^2} - (x-3)^2 t^5 \frac{\partial u}{\partial t} = 0.$$

Note: Do **not** solve these ordinary differential equations.
Solution:

$$\begin{aligned}
u(x,t) &= X(x)T(t) \\
\frac{\partial^2 u}{\partial x^2} &= X'' \bullet T \\
\frac{\partial^2 u}{\partial t^2} &= X \bullet T'' \\
e^{x-t}X''T - (x-3)^2 t^5 XT'' &= 0 \\
e^x e^{-t} X''T &= (x-3)^2 t^5 XT'' \\
\frac{e^x X''}{(x-3)^2 X} &= \frac{t^5 e^t T''}{T} = -\lambda
\end{aligned}$$

The last step is the observation that one side is a function only of x and the other side is a function only of t so they must be constant. Any name for the constant may be used. I chose $-\lambda$ since the next step of often an eigenvalue problem. Taking one at a time produces the two O.D.E.s.

$$\begin{aligned}
e^x X'' + \lambda(x-3)^2 X &= 0 \\
t^5 e^t T'' + \lambda T &= 0.
\end{aligned}$$

b.) (15 pts.) Find

$$\mathcal{L}^{-1}\left\{\frac{2s^3 + 5s^2 + 6s + 7}{\left(s^2 - 1\right)\left(s^2 + 4s + 5\right)}\right\}.$$

Solution: After completing the square in the quadratic factor in the denominator, we set up the partial fractions expansion needed.

$$\begin{aligned}
\frac{2s^3 + 5s^2 + 6s + 7}{\left(s^2 - 1\right)\left(s^2 + 4s + 5\right)} &= \frac{2s^3 + 5s^2 + 6s + 7}{\left(s^2 - 1\right)\left(s^2 + 4s + 4 + 1\right)} \\
&= \frac{2s^3 + 5s^2 + 6s + 7}{(s+1)(s-1)\left[(s+2)^2 + 1\right]} \\
&= \frac{A}{s+1} + \frac{B}{s-1} + \frac{C(s+2) + D}{(s+2)^2 + 1}
\end{aligned}$$

The numerator of the second fraction could be $Bs + C$, but that would require some extra algebra to invert the Laplace transform.

We multiply by the common denominator.

$$2s^3+5s^2+6s+7 = A(s-1)\left[(s+2)^2+1\right]+B(s+1)\left[(s+2)^2+1\right]+[C(s+2)+D](s-1)(s+1)$$

Set $s=-1$.

$$-2+5-6+7 = 4 = A(-2)(2)$$
$$A = -1$$

Set $s=1$.

$$2+5+6+7 = 20 = B(2)(10)$$
$$B = 1$$

Set $s=-2$.

$$-16+20-12+7 = -1 = A(-3)+B(-1)+D(-3)(-1)$$
$$-1 = 3-1+3D$$
$$D = -1$$

Equate the coefficients of s^3.

$$2 = A+B+C$$
$$C = 2$$

Thus

$$\frac{2s^3+5s^2+6s+7}{\left(s^2-1\right)\left(s^2+4s+5\right)} = \frac{-1}{s+1}+\frac{1}{s-1}+\frac{2(s+2)-1}{(s+2)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{2s^3+5s^2+6s+7}{\left(s^2-1\right)\left(s^2+4s+5\right)}\right\} = -e^{-t}+e^{t}+2e^{-2t}\cos t-e^{-2t}\sin t$$

5. (a) (15 pts.) Find the first five non-zero terms of the Fourier *sine* series for the function

$$f(x) = \begin{cases} 0 & 0 < x < \pi \\ 1 & \pi < x < 2\pi \end{cases}$$

Solution:

$$f(x) = \sum_{k=1}^{\infty} \alpha_k \sin \frac{k\pi x}{L}$$

where

$$\alpha_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}dx, \quad k = 1,2,3,\ldots$$

Here $L = 2\pi$ so

$$f(x) = \sum_{k=1}^{\infty} \alpha_k \sin\left(\frac{kx}{2}\right)$$

where

$$\alpha_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin\left(\frac{kx}{2}\right)dx, \quad k = 1,2,3,\ldots$$

Thus

$$\begin{aligned} a_k &= \frac{1}{\pi}\left[\int_0^{\pi} 0 \cdot \sin\left(\frac{kx}{2}\right)dx + \int_{\pi}^{2\pi} 1 \cdot \sin\left(\frac{kx}{2}\right)dx\right] \\ &= \left(\frac{1}{\pi}\right)\left(\frac{-2}{k}\right)\left[\cos\left(\frac{kx}{2}\right)\right]_{\pi}^{2\pi} \\ &= \frac{-2}{k\pi}\left[\cos k\pi - \cos\left(\frac{k\pi}{2}\right)\right] \end{aligned}$$

Calculating until we have five that are non zero, we obtain

$$\begin{aligned} a_1 &= \frac{-2}{\pi}[-1-0] = \frac{2}{\pi} \\ a_2 &= \frac{-2}{2\pi}[1+1] = \frac{-4}{2\pi} \\ a_3 &= \frac{-2}{3\pi}[-1-0] = \frac{2}{3\pi} \\ a_4 &= \frac{-2}{4\pi}[1-1] = 0 \\ a_5 &= \frac{-2}{5\pi}[-1-0] = \frac{2}{5\pi} \\ a_6 &= \frac{-2}{6\pi}[1+1] = \frac{-4}{6\pi} \end{aligned}$$

Finally, the Fourier series is

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \alpha_k \sin\frac{kx}{2} = a_1 \sin\left(\frac{x}{2}\right) + a_2\sin\left(\frac{2x}{2}\right) + a_3\sin\left(\frac{3x}{2}\right) + \ldots \\ &= \frac{2}{\pi}\sin\left(\frac{x}{2}\right) - \frac{4}{2\pi}\sin\left(\frac{2x}{2}\right) + \frac{2}{3\pi}\sin\left(\frac{3x}{2}\right) + \frac{2}{5\pi}\sin\left(\frac{5x}{2}\right) - \frac{4}{6\pi}\sin\left(\frac{6x}{2}\right) + \ldots \end{aligned}$$

5(b) (10 pts.) To what value does the Fourier series of 5a converge at each of the following points?
Solution:
(i) $x = -\frac{3\pi}{2}$ $\quad f\left(-\frac{3\pi}{2}\right) = -1$ $\qquad$ (ii) $x = 0$ $\quad f(0) = 0$ $\qquad$ (iii) $x = \pi$ $\quad f(\pi) = \frac{1}{2}$

(iv) $x = \frac{3\pi}{2}$ $\quad f\left(\frac{3\pi}{2}\right) = 1$ $\quad$ (v) $x = \frac{5\pi}{2}$ $\quad f\left(\frac{5\pi}{2}\right) = -1.$

6 (25 pts.) Solve the following initial-boundary value problem.

$$\begin{array}{ll} \text{PDE} & u_t = 3u_{xx}, \qquad 0 < x < 4, \quad t > 0 \\ \text{BCs} & u_x(0,t) = 0 \qquad u_x(4,t) = 0 \\ \text{IC} & u(x,0) = \cos\left(\frac{\pi}{2}x\right) - 7\cos\left(\frac{3\pi}{4}x\right) + 5\cos\left(\frac{3\pi}{2}x\right) \end{array}$$

You must derive the solution. Your solution should not have any arbitrary constants in it. Show **all** steps.

Solution: Separation of Variables:

$$u(x,t) = X(x)T(t)$$
$$X(x)T'(t) = 3X''(x)T(t)$$
$$\frac{X''}{X} = \frac{T'}{3T} = -\lambda$$

Two ordinary differential equations result.

$$X'' + \lambda X = 0$$
$$T' + 3\lambda T = 0$$

The boundary conditions lead to boundary conditions on X.

$$u_x(0,t) = X'(0)T(t) = 0 \quad \Rightarrow X'(0) = 0$$
$$u_x(4,t) = X'(4)T(t) = 0 \quad \Rightarrow X'(4) = 0$$

We next solve the resulting eigenvalue problem. The characteristic equation gives $r = \pm\sqrt{-\lambda}$. We look at the discriminant being positve, zero or negative.

Case 1. -$\lambda > 0 \quad -\lambda = \mu^2$.

$$X = c_1 e^{x} + c_2 e^{-\mu x}$$
$$X' = \mu(c_1 e^{x} - c_2 e^{-\mu x})$$
$$X'(0) = \mu(c_1 - c_2) = 0$$
$$c_1 = c_2$$
$$X'(4) = \mu c_1\left(e^{4\mu} - e^{-4\mu}\right) = 0$$
$$c_1 = c_2 = 0$$

Case 2 $\quad -\lambda = 0$

$$X = c_1 + c_2 x$$
$$X' = c_2$$
$$X'(0) = c_2 = 0$$
$$X'(4) = c_2 = 0$$

So $\lambda = 0$ is an eigenvalue and we will lable th corresponding eigenfunction

$$X_0 = c_0.$$

Case 3 $-\lambda < 0 \quad -\lambda = -\mu^2$.

$$\begin{aligned}
X &= c_1 \cos \mu x + c_2 \sin \mu x \\
X' &= \mu(-c_1 \sin \mu x + c_2 \cos \mu x) \\
X'(0) &= \mu c_2 = 0 \\
c_2 &= 0 \\
X'(4) &= -c_1 \mu \sin 4\mu = 0
\end{aligned}$$

So, non-zero solutions require $\sin 4\mu = 0$. We have

$$\begin{aligned}
4\mu &= n\pi && n = 1,2,3,\ldots \\
\mu_n &= \frac{n\pi}{4} && n = 1,2,3,\ldots \\
\lambda_n &= \left(\frac{n\pi}{4}\right)^2 && n = 1,2,3,\ldots \\
X_n &= c_n \cos\left(\frac{n\pi}{4}x\right) && n = 1,2,3,\ldots
\end{aligned}$$

We can combine cases 2 and 3 by adjusting the range of the index.

$$\begin{aligned}
\mu_n &= \frac{n\pi}{4} && n = 0,1,2,3,\ldots \\
\lambda_n &= \left(\frac{n\pi}{4}\right)^2 && n = 0,1,2,3,\ldots \\
X_n &= c_n \cos\left(\frac{n\pi}{4}x\right) && n = 0,1,2,3,\ldots
\end{aligned}$$

The d.e. for T.

$$\begin{aligned}
T' + 3\lambda T &= 0 \\
T' + 3\left(\frac{n\pi}{4}\right)^2 T &= 0 \\
T_n &= A_n \exp\left(-3\left(\frac{n\pi}{4}\right)^2 t\right)
\end{aligned}$$

We combine the results.

$$\begin{aligned}
u_n(x,t) &= X_n(x)T_n(t) \\
&= A_n c_n \exp\left(-3\left(\frac{n\pi}{4}\right)^2 t\right)\cos\left(\frac{n\pi}{4}x\right)
\end{aligned}$$

A formal solution is obtained by summing. (The two constants are combined in this step.)

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\
&= \sum_{n=0}^{\infty} a_n \exp\left(-3\left(\frac{n\pi}{4}\right)^2 t\right)\cos\left(\frac{n\pi}{4}x\right)
\end{aligned}$$

To find the coefficients, we use the initial condition.

$$\begin{aligned}
u(x,0) &= \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{4}x\right) \\
&= \cos\left(\frac{\pi}{2}x\right) - 7\cos\left(\frac{3\pi}{4}x\right) + 5\cos\left(\frac{3\pi}{2}x\right)
\end{aligned}$$

Matching terms leads to $a_2 = 1$, $a_3 = -7$ and $a_6 = 5$. All the rest are zero. With this, the solution is

$$u(x,t) = \exp\left(-3\left(\frac{2\pi}{4}\right)^2 t\right)\cos\left(\frac{2\pi}{4}x\right) - 7\exp\left(-3\left(\frac{3\pi}{4}\right)^2 t\right)\cos\left(\frac{3\pi}{4}x\right) + 5\exp\left(-3\left(\frac{6\pi}{4}\right)^2 t\right)\cos\left(\frac{6\pi}{4}x\right)$$

7. (a) (13 pts.) Find a general solution of

$$y'' + 2y' + y = \frac{e^{-x}}{x^2}$$

Solution: We solve this d.e. by the method of variation of parameters. The characteristic equation is

$$r^2 + 2r + 1 = (r+1)^2 = 0.$$

Hence

$$y_h = c_1 e^{-x} + c_2 x e^{-x}.$$
$$y_1 = e^{-x} \qquad y_1' = -e^{-x}$$
$$y_2 = xe^{-x} \qquad y_2' = (1-x)e^{-x}$$

Assuming

$$y_p = v_1 y_1 + v_2 y_2$$

we have two equations for v_1' and v_2'.

$$e^{-x} v_1' + x e^{-x} v_2' = 0 \qquad \text{(A)}$$
$$-e^{-x} v_1' + (1-x) e^{-x} v_2' = \frac{e^{-x}}{x^2} \qquad \text{(B)}$$

Add these to obtain

$$e^{-x} v_2' = \frac{e^{-x}}{x^2} \qquad \text{(C)}$$
$$v_2' = \frac{1}{x^2}$$
$$v_2 = \frac{-1}{x} + c_2$$

Now insert equation (C) into equation (A) to obtain

$$e^{-x} v_1' + x \frac{e^{-x}}{x^2} = 0$$
$$v_1' = \frac{-1}{x}$$
$$v_1 = -\ln x + c_1$$

A general solution is

$$y = (c_1 - \ln x) e^{-x} + \left(c_2 - \frac{1}{x}\right) x e^{-x}.$$

7 (b) (12 pts.) Find the power series solution to

$$y'' + xy' - 2y = 0$$

near $x = 0$. Be sure to give the recurrence relation for the coefficients of the power series. Indicate the two linearly independent solutions and give the first six nonzero terms of the solution.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

so

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

.

The differential equation $\Rightarrow$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

In the first series, we set $k = n - 2$ (which is the same as $n = k + 2$).

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Next, replace n by k in the other two series.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} a_k k x^k - \sum_{k=0}^{\infty} 2a_k x^k = 0$$

Observe that the middle series has one less term. We bring out the $k = 0$ terms from the first and last series and combine the rest.

$$(2a_2 - 2a_0) + \sum_{k=1}^{\infty} [(k+2)(k+1) a_{k+2} + (k-2) a_k] x^k = 0$$

From the first term

$$a_2 = a_0$$

From the rest, we obtain the recurrence relation.

$$(k+2)(k+1) a_{k+2} + (k-2) a_k = 0 \qquad k = 1, 2, 3, \ldots$$

$$a_{k+2} = \frac{2-k}{(k+2)(k+1)} a_k \qquad k = 1, 2, 3, \ldots$$

We have three non-zero coefficients (a_o, a_1, a_2) so we need three more.

$$\begin{aligned}
k &= 1 \quad \Rightarrow a_3 = \frac{1}{3 \cdot 2} a_1 \\
k &= 2 \quad \Rightarrow a_4 = 0 \\
k &= 3 \quad \Rightarrow a_5 = \frac{-1}{5 \cdot 4} a_3 = \frac{-1}{5!} a_1 \\
k &= 4 \quad \Rightarrow a_6 = \frac{-2}{6 \cdot 5} a_4 = 0 \\
k &= 5 \quad \Rightarrow a_7 = \frac{-3}{7 \cdot 6} = \frac{(-1)^2 3 \cdot 1}{7!} a_1
\end{aligned}$$

Now, the solution is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= a_0 \left(1 + x^2\right) + a_1 \left(x + \frac{1}{3!} x^3 - \frac{1}{5!} x^5 + \frac{3 \cdot 1}{7!} x^7 + \ldots\right)
\end{aligned}$$

8 (a) (15 pts.) Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0 \qquad 0 < x < 1$$
$$y'(0) = y(1) = 0$$

Be sure to consider the cases $\lambda < 0, \lambda = 0,$ and $\lambda > 0.$
Solution: The characteristic equation is $r^2 + \lambda = 0$. Thus $r = \pm\sqrt{-\lambda}$. We consider the three cases of the quantity under the radical being positive, zero or negative.
Case 1. $-\lambda > 0$. We write $-\lambda = \mu^2$. The solution to the d.e is

$$y = c_1 e^x + c_2 e^{-\mu x}$$
$$y' = \mu(c_1 e^x - c_2 e^{-\mu x})$$

From the boundary conditions,1

$$y'(0) = \mu(c_1 - c_2) = 0$$
$$c_1 = c_2$$
$$y(1) = c_1\left(e + \frac{1}{e}\right) = 0$$
$$c_1 = c_2 = 0$$

There is no non-zero solution in this case.
Case 2 $\quad -\lambda = 0 \quad$ The solution to the d.e is

$$y = c_1 + c_2 x$$
$$y' = c_2$$

From the boundary conditions,

$$y'(0) = c_2 = 0$$
$$y(1) = c_1 = 0$$

Again, there is no non-zero solution.
Case 3. $-\lambda < 0$ We write $-\lambda - -\mu^2$. $\quad r = \pm\sqrt{-\mu^2} = \pm\mu i$. The solution to the d.e. is

$$y = c_1 \cos \mu x + c_2 \sin \mu x$$
$$y' = \mu(-c_1 \sin \mu x + c_2 \cos \mu x)$$

From the boundary conditions,

$$y'(0) = \mu c_2 = 0$$
$$c_2 = 0$$
$$y(1) = c_1 \cos \mu$$

For a non-zero solution, we must have

$$\cos \mu = 0$$
$$\mu_n = (2n+1)\frac{\pi}{2} \qquad n = 0, 1, 2, \ldots$$

So the eigenvalues (λ_n) and corresponding eigenfunctions (y_n) are

$$\lambda_n = \mu_n^2 = \left[(2n+1)\frac{\pi}{2}\right]^2 \qquad n = 0, 1, 2, \ldots$$
$$y_n = c_n \cos\left(\frac{2n+1}{2}\pi x\right) \qquad n = 0, 1, 2, \ldots$$

8(b) (10 pts.) Solve the initial value problem

$$\frac{dy}{dx} + y\tan x = \frac{\sec x}{y^2} \quad y(0) = 1$$

Solution: This is a Bernoulli equation. We write it as

$$y^2\frac{dy}{dx} + (\tan x)y^3 = \sec x$$

Let

$$v = y^3$$

$$\frac{dv}{dx} = 3y^2\frac{dy}{dx}$$

The d.e becomes

$$\frac{1}{3}\frac{dv}{dx} + \tan xv = \sec x$$

$$\frac{dv}{dx} + 3\tan xv = 3\sec x$$

This is a linear d.e. The integrating factor is

$$\mu = e^{\int 3\tan x dx} = e^{-3\ln\cos x} = e^{\ln(\cos x)^{-3}}$$

$$= (\cos x)^{-3} = \sec^3 x$$

Multiply by the integrating factor.

$$\sec^3 x\frac{dv}{dx} + 3\tan x\sec^3 xv = 3\sec^4 x$$

$$\frac{d}{dx}\left(v\sec^3 x\right) = 3\sec^4 x$$

$$\left(v\sec^3 x\right) = \tan x + \frac{1}{3}\tan^3 x + C$$

Multiply by $\cos^3 x$.

$$v = y^3 = \sin x\cos^2 x + \frac{1}{3}\sin^3 x + C\cos^3 x$$

From the initial condition

$$1 = C$$

The impicit solution is

$$y^3 = \sin x\cos^2 x + \frac{1}{3}\sin^3 x + \cos^3 x$$

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int x \cos bx dx = \frac{1}{b^2}(\cos bx + bx \sin bx) + C$
$\int x \sin bx dx = \frac{1}{b^2}(\sin bx - bx \cos bx) + C$
$\int \tan u du = -\ln(\cos u) + C$
$\int \tan^2 u du = \tan u - u + C$
$\int \sec u du = \ln(\sec u + \tan u) + C$
$\int \sec^2 u du = \tan u + C$
$\int \sec^3 u du = \frac{1}{2}[\sec u \tan u + \ln(\sec u + \tan u] + C$
$\int \sec^4 u du = \tan u + \frac{1}{3}\tan^3 u + C$
$\int \ln u du = u \ln u - u + C$
$\int u \ln u du = \frac{1}{2}u^2 \ln u - \frac{1}{4}u^2 + C$
$\int u^2 \ln u du = \frac{1}{3}u^3 \ln u - \frac{1}{9}u^3 + C$
$\int \frac{\ln u}{u} du = \frac{1}{2}\ln^2 u + C$