Ma 221

Chapter 1 - Basic Concepts

Classification of Differential Equations

A differential equation is an equation involving an unknown function and one or more of its derivatives. Thus it is a relation of the form

\[ F(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}) = 0 \]

\( F \) is given, and we are to find \( y \). The above is an ordinary differential equation of order \( n \).

Example

\[ \frac{dy}{dx} = f(x) \text{ or } y' = f(x) \]

Definition. The order of a differential equation is the order of the highest derivative appearing in the equation. If the equation is a polynomial in the unknown function and its derivatives, then the degree of such an equation is the power to which the highest derivative is raised.

Example

\[ a(x)y'' + b(x)y' + c(x)y = f(x) \]. second order, first degree

Remark: \( f(x) = 0 \Rightarrow \text{homogeneous equation. } f(x) \neq 0 \text{ nonhomogeneous.} \)

Example

\[ (\frac{d^3 y}{dt^3})^2 + 5s^4 t^3 = 0 \]. 3rd order, 2nd degree

Example

\[ 2xy'' - (x + 3)y' + 6x^4y = 0 \]. 2nd order, first degree

Up until now we have mentioned only ordinary differential equations. We shall eventually be concerned with partial differential equations also.

Example

\[ uu_{xx} = \frac{1}{c^2}u_t \]

Here \( u = u(x,t) \). This is a partial differential equation. Here \( c \) is a constant.

Solutions of Differential Equations.

Consider the \( n \)-th order ordinary differential equation

\[ F(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}) = 0. \quad (1) \]

Definition. A solution of the ordinary differential equation (1) is a real valued function \( y(x) \) defined on some interval \( I \) such that:

1. \( y(x) \) and its first \( n \) derivatives exist for each \( x \in I \).
2. The substitution of \( y(x) \) into the differential equation makes the equation an identity in the interval \( I \).

Remarks: I may be \((-\infty, \infty), [a,b], (a,b), [a,b], (a,b). We assume I is not degenerate. Now \( I \Rightarrow y \) and its \( n - 1 \) first derivatives are continuous.

Example

\[ y' = x \quad -\infty < x < \infty \quad y = \frac{1}{2}x^2 \] is a solution since

\[ \frac{dy}{dx} = \frac{d}{dx} \left( \frac{1}{2}x^2 \right) = x \]
Thus \( y(x) = \frac{1}{2} x^2 \) is solution for \(-\infty < x < \infty\).

The function

\[
y = \begin{cases} 
\frac{1}{2} x^2 - 1 & x \geq 0 \\
\frac{1}{2} x^2 & x < 0 
\end{cases}
\]

is not a solution of the differential equation on \(-\infty < x < \infty\) due to the discontinuity at \(x = 0\). \( y = \frac{1}{2} x^2 - 1 \) is a solution on \(0 < x < \infty\) whereas \( y = \frac{1}{2} x^2 \) is a solution on \(-\infty < x < 0\).

**Example** \( y' + y = 0 \) One solution is \( y = e^{-x} \). The general solution is \( y = ce^{-x} \), where \( c \) is any constant.

**Remarks about solutions:**

1. Sometimes we obtain the solution to a differential equation implicitly in the form \( f(x, y) = 0 \). We need not always solve for \( y \) as a function of \( x \) (cannot). However, we can verify that we have a solution by implicit differentiation.

   **Example** \( e^y \frac{dy}{dx} + x = 0 \)
   \[
   \Rightarrow e^y \frac{dy}{dx} + x \, dx = 0 \\
   \Rightarrow e^y + \frac{x^2}{2} = c \quad (\ast)
   \]
   We could write \( y = \ln(c - \frac{x^2}{2}) \) but need not. To see if \((\ast)\) is a solution we differentiate implicitly. \((\ast)\)
   \[
   \Rightarrow e^y \frac{dy}{dx} + x = 0.
   \]

2. Not all equations have solutions.

   **Example** \( (y')^2 + y^2 = -1 \) has no solution.
   Clearly \( y = 0 \) is not a solution. If \( y \neq 0 \), \( \Rightarrow y^2 > 0 \) and \((y')^2 > 0\).

**Initial and Boundary Value Problems**

We have seen above that a differential equation need not have a unique solution.

**Example** \( y' = x \quad y = \frac{1}{2} x^2 + c \).

If we are given some subsidiary condition then we will “pick” out a unique solution. For example, if we are given the initial conditions \( y(0) = -1 \Rightarrow c = -1 \) \( \Rightarrow y = \frac{1}{2} x^2 - 1 \).

For first order equations one is given one condition. For second order equations one needs two conditions.

**Example** \( y'' + y = 0 \)

One may verify directly that \( y = c_1 \sin x + c_2 \cos x \) is the solution, where \( c_1 \) and \( c_2 \) are constants.

If, for example, we are given \( y(0) = 0 \) and \( y'(0) = 1 \) \( \Rightarrow y(0) = c_1 \sin 0 + c_2 \cos 0 = c_2 = 0 \Rightarrow y = c_1 \sin x \Rightarrow y'(x) = c_1 \cos x \Rightarrow y'(0) = c_1 = 1 \)

Thus \( y = \sin x \) is the solution.

We could have been given the boundary conditions \( y(0) = 0 \quad \Rightarrow c_2 = 0 \) as before. Also \( y\left(\frac{\pi}{2}\right) = c_1 \sin \frac{\pi}{2} = 2 \quad \Rightarrow c_1 = 2 \Rightarrow y = 2 \sin x \)

The above are two different kinds of conditions. When the two conditions are given at the *same* point, they are called Initial Conditions; when the two conditions are given at two *different* points, they are called Boundary Conditions.

The equation together with the two conditions is called either an Initial Value Problem (I.V.P.) or a
Boundary Value Problem (B.V. P.).

**Example**

DE $y'' = 2x$

B.C. $y(0) = 0 \quad y(2) = 1$

This is a B.V.P.

$y' = x^2 + c_1$

so

$y = \frac{x^3}{3} + c_1x + c_2$

$y(0) = 0 \Rightarrow c_2 = 0$

$y(2) = 1 \Rightarrow \frac{8}{3} + 2c_1 = 1 \Rightarrow 2c_1 = 1 - \frac{8}{3} = -\frac{5}{3}$ and therefore $c_1 = -\frac{5}{6}$

$y(x) = \frac{x^3}{3} - \frac{5}{6}x$

is the solution.

**Example**

D.E. $y'' = 2x$

I.C. $y(1) = 0 \quad y'(1) = -1$

This is an I.V.P.

$y' = x^2 + c_1$

so $y(x) = \frac{x^3}{3} + c_1x + c_2$

$y(1) = 0 \quad \Rightarrow \frac{1}{3} + c_1 + c_2 = 0$

$y'(1) = -1 \quad \Rightarrow 1 + c_1 = -1$

$\Rightarrow c_1 = -2 \quad \text{and} \quad \frac{1}{3} - 2 + c_2 = 0 \Rightarrow c_2 = \frac{5}{3}$

Thus

$y(x) = \frac{x^3}{3} - 2x + \frac{5}{3}$

is the solution.