Ma 221 Final Exam Review

Problems on Fourier Series, Boundary Value Problems and Separation of Variables

Ma 227 Final Exam 95S

Problem 1

Find the first four nonzero terms of the Fourier sine series of

$$f(x) = \begin{cases} 0 & 0 < x < \pi \\ -2 & \pi < x < 2\pi \end{cases}$$

Solution:

If $f(x)$ is a function defined on $[0, L]$, then its Fourier sine expansion is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where} \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Here $L = 2\pi$ so that $f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{nx}{2}\right)$ and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin\frac{nx}{2}\,dx$$

Hence $a_n = \frac{1}{\pi}\left[\int_0^{\pi} 0\sin\frac{nx}{2}\,dx + \int_{\pi}^{2\pi}(-2)\sin\frac{nx}{2}\,dx\right] =$

$$\frac{1}{\pi}(4)\left[\frac{\cos\pi n - \cos(\frac{1}{2}\pi n)}{n}\right].$$

Evaluating this last expression for $n = 1, 2, 3, 4, 5$ we get

$n = 1 \qquad a_1 = \frac{4}{\pi}[-1]$

$n = 2 \qquad a_2 = \frac{4}{\pi}[1]$

$n = 3 \qquad a_3 = \frac{4}{\pi}\left[-\frac{1}{3}\right] = -\frac{4}{3\pi}$

$n = 4 \qquad a_4 = 0$

$n = 5 \qquad a_5 = \frac{4}{\pi}[-\frac{1}{5}] = -\frac{4}{5\pi}$

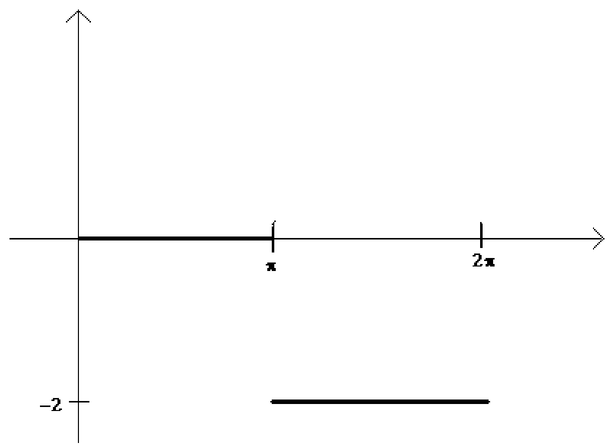
Thus $f(x) = -\frac{4}{\pi}\sin\frac{x}{2} + \frac{4}{\pi}\sin x - \frac{4}{\pi}\sin\frac{3x}{2} + 0\sin 2x - \frac{4}{\pi}\sin\frac{5x}{2} + \cdots$

b) (8 points)
Sketch the graph of the function to which the Fourier sine series of the function

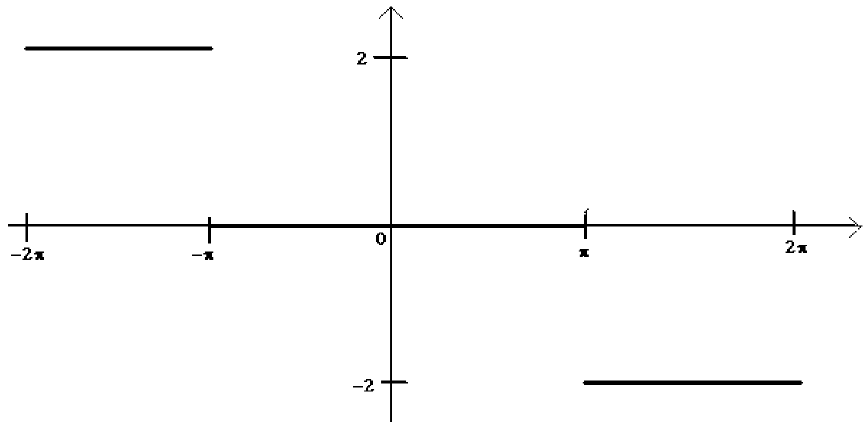
$$f(x) = \begin{cases} 0 & 0 < x < \pi \\ -2 & \pi < x < 2\pi \end{cases}$$

converges on $-2\pi < x < 4\pi$.

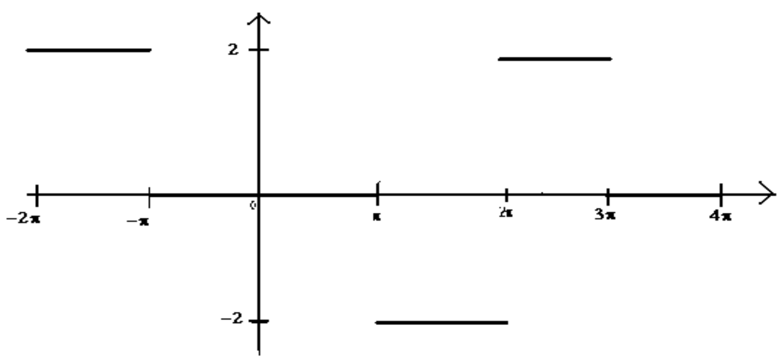
Solution
The graph of the given function is below.



Since we were asked to find the Fourier sine expansion of $f(x)$, this means that we are seeking an odd expansion of f. Hence the graph above is reflected first across the $y - axis$, and then across the $x - axis$ to get an odd function. The result is given below.



The Fourier sine series generates an odd function with period $2L$. Here $L = 2\pi$, so the function generated by the Fourier series has period $2(2\pi) = 4\pi$. Since the last graph above given the function on the interval $[-2\pi, 2\pi]$, i.e., on an interval of length 4π, we may move this graph either to the left or the right to get the function anywhere. Thus we have



c) (9 points)
Find the eigenvalues and eigenfunctions for the problem

$$y'' + \lambda y = 0, \quad y'(0) = y'(1) = 0$$

Be sure to check the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.
Solution
I. Consider the case $\lambda < 0$ first. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. The DE becomes

$y'' - \alpha^2 y = 0.$

The general solution of this equation is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$ Thus

$y'(x) = c_1 \alpha e^{\alpha x} - c_2 \alpha e^{-\alpha x}$

$y'(0) = c_1\alpha - c_2\alpha = 0$ and $y'(1) = c_1 \alpha e^{\alpha} - c_2 \alpha e^{-\alpha} = 0.$

The first equation implies that $c_1 = c_2$. Thus the second equation becomes $c_1(e^{\alpha} - e^{-\alpha}) = 0.$ Thus $c_1 = 0$, this tells us that $c_2 = 0$ also. Therefore $y = 0$ is the only solution if $\lambda < 0.$

II. Suppose $\lambda = 0.$ The DE becomes $y'' = 0$ which has the solution $y = c_1 x + c_2.$ The boundary conditions imply $c_1 = 0,$ so that $y = c_2.$ Thus $y = c_2$ where $c_2 \neq 0$ is an eigenfunction corresponding to the eigenvalue $\lambda = 0.$

III. Suppose $\lambda > 0.$ Let $\lambda = \beta^2$ where $\beta \neq 0.$ The DE becomes

$y'' + \beta^2 y = 0.$

The general solution of this equation is $y(x) = c_1 \sin \beta x + c_2 \cos \beta x.$ Thus

$y'(x) = c_1 \beta \cos \beta x - c_2 \beta \sin \beta x$

Now $y'(0) = c_1 \beta = 0$ Since $\beta \neq 0,$ we must have $c_1 = 0.$ Thus $y(x) = c_2 \cos \beta x.$ Now $y'(x) = -c_2 \beta \sin \beta x$ and $y'(1) = -c_2 \beta \sin \beta = 0.$ For a nontrivial solution we must have $c_2 \neq 0.$ This means that $\sin \beta = 0$ or $\beta = n\pi,$ $n = 1, 2, 3, \ldots$ The eigenvalues are therefore $\lambda = \beta^2 = n^2\pi^2$ and the corresponding eigenfunctions are $y_n = a_n \cos n\pi x,$ $n = 1, 2, 3, \ldots$

We may also include the eigenfunction found in II above by allowing n to equal 0. Hence all of the eigenfunctions are given by $y_n = a_n \cos n\pi x,$ $n = 0, 1, 2, 3, \ldots$ with corresponding eigenvalues $\lambda = n^2\pi^2,$ $n = 0, 1, 2, 3, \ldots$

Problem 2

a) (10 points)

Use separation of variables, $u(x,t) = X(x)T(t)$, to find ordinary differential equations which $X(x)$ and $T(t)$ must satisfy if $u(x,t)$ is to be a solution of

$$11t^2 x^9 u_{xx} - (t-3)(x+2) u_{ttt} = 0$$

Solution:

$u_x = X'T, \qquad u_{xx} = X''T, \qquad\qquad u_t = XT'$, etc.

Thus the given equation becomes

$11t^2x^9X''T - (t-3)(x+2)XT''' = 0$

$\Rightarrow$

$$11x^9\frac{X''}{(x+2)X} = (t-3)\frac{T'''}{t^2T} = k, \qquad k \text{ a constant}$$

This yields the two ODEs

$$11x^9X'' - k(x+2)X = 0$$
$$(t-3)T''' - kt^2T = 0$$

b) (15 points)

Solve:

P.D.E.: $u_{xx} - 4u_{tt} = 0$

B.C.'s: $u_x(0,t) = 0 \qquad u_x(\pi,t) = 0$

I.C.'s: $u(x,0) = 0 \qquad u_t(x,0) = -8\cos(4x) + 17\cos(8x)$

Solution:

Let $u(x,t) = X(x)T(t)$. Then differentiating and substituting in the PDE yields

$X''T = 4XT''$

$\Rightarrow \qquad \dfrac{X''}{X} = 4\dfrac{T''}{T}$

Using the argument that the left hand side is purely a function of x and the right hand side is purely a function of t, and the only way that they can be equal is if they are equal to a constant, we get

$\dfrac{X''}{X} = 4\dfrac{T''}{T} = k \qquad k$ a constant

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T'' - \frac{1}{4}kT = 0$$

The boundary condition $u_x(0,t) = 0$ implies, since $u_x(x,t) = X'(x)T(t)$ that

$X'(0)T(t) = 0.$ We cannot have $T(t) = 0,$ since this would imply that $u(x,t) = 0$. Thus $X'(0) = 0.$ Similarly, the boundary condition $u_x(\pi,t) = 0$ leads to $X'(\pi) = 0.$

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X'(0) = X'(\pi) = 0$$

This boundary value problem is very similar to the one given in Problem 1(c) above. (Its solution was discussed in the slide show Eigenvalues and Eigenfunctions for Boundary Value Problems.) The solution is

$$k = -n^2 \qquad X_n(x) = a_n \cos nx \qquad n = 1,2,3,\ldots$$

Substituting the values of k into the equation for $T(t)$ leads to

$$T'' + \frac{n^2}{4}T = 0$$

which has the solution $T_n(t) = b_n \sin\frac{nt}{2} + c_n \cos\frac{nt}{2},\ n = 1,2,3,\ldots$

The initial condition $u(x,0) = 0$ implies $X(x)T(0) = 0$ so that $T(0) = 0.$ Thus $c_n = 0.$

We now have the solutions

$$u_n(x,t) = A_n \cos nx \sin\frac{nt}{2} \qquad n = 1,2,3,\ldots$$

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \cos nx \sin\frac{nt}{2}$$

satisfies the PDE, the boundary conditions, and the first initial condition. Since

$$u_t(x,t) = \sum_{n=1}^{\infty} A_n\left(\frac{n}{2}\right)\cos nx \cos\frac{nt}{2}$$

The last initial condition leads to

$$u_t(x,0) = -8\cos(4x) + 17\cos(8x) = \sum_{n=1}^{\infty} A_n\left(\frac{n}{2}\right)\cos nx.$$

Matching the cosine terms on both sides of this equation leads to

$A_4\left(\frac{4}{2}\right) = -8$ so that $A_4 = -4$ and $A_8\left(\frac{8}{2}\right) = 17$ so that $A_8 = \frac{17}{4}$. All of the other constants must be zero, since there are no cosine terms on the left to match with. Thus

$$u(x,t) = -4\cos 4x \sin\frac{4t}{2} + \frac{17}{4}\cos 8x \sin\frac{8t}{2} = -4\cos 4x \sin 2t + \frac{17}{4}\cos 8x \sin 4t$$

Ma 227 Final Exam 97F

1. a) Find the first four non-zero terms on the Fourier cosine series of

$$f(x) = \begin{cases} 3 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

Cosine Formula: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$a_0 = 1\left[\int_0^1 3dx + \int_1^2 0dx\right] = 3$$

$$a_n = \int_0^1 3\cos\frac{n\pi x}{2}dx + \int_1^2 0\cos\frac{n\pi x}{2}dx = \frac{6}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^1 = \frac{6}{n\pi}\sin\frac{n\pi}{2}$$

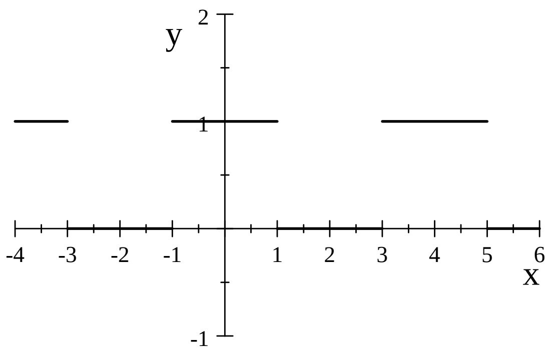
$$a_n = \begin{cases} \frac{6}{n\pi}(-1)^{\frac{n-1}{2}} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Thus $f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2} = \sum_{n=1}^{9} \frac{6(-1)^{n+1}}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}$

computing the first few terms:

$$f(x) = \frac{3}{2} + \frac{6}{\pi}\cos\frac{1}{2}\pi x - \frac{2}{\pi}\cos\frac{3}{2}\pi x + \frac{6}{5\pi}\cos\frac{5}{2}\pi x - \frac{6}{7\pi}\cos\frac{7}{2}\pi x + \frac{2}{3\pi}\cos\frac{9}{2}\pi x$$

1. b) Sketch the graph of $f(x)$ on $-4 < x < 6$



1. c) Solve the boundary value problem:

$$y''(x) - y(x) = x; \qquad y(0) = 0; \qquad y'(1) = 1$$

homogeneous solution: $y''(x) - y(x) = 0$

characteristic equation: $r^2 - 1 = 0 \Rightarrow r = \pm 1$

$$y(x) = c_1 e^x + c_2 e^{-x}$$

particular solution: $\left.\begin{array}{l} y(x) = Ax + B \\ y'(x) = A \\ y''(x) = 0 \end{array}\right\} \Rightarrow A = -1 \Rightarrow y(x) = -x$

general solution: $y(x) = c_1 e^x + c_2 e^{-x} - x$ then $y'(x) = c_1 e^x - c_2 e^{-x} - 1$

B.C. $\Rightarrow$ $y(0) = c_1 e^0 + c_2 e^{-0} - 0 = 0 \Rightarrow c_1 = -c_2$

and $y'(1) = c_1 e^1 - c_2 e^{-1} - 1 = 1 \Rightarrow c_1 e - c_2 e^{-1} = 2$

$\begin{array}{l} c_1 = -c_2 \\ c_1 e - c_2 e^{-1} = 2 \end{array}$, Solution is : $\left\{c_2 = -\frac{2}{e+e^{-1}}, c_1 = \frac{2}{e+e^{-1}}\right\}$,

So

$$y(x) = \frac{2}{e+e^{-1}} e^x - \frac{2}{e+e^{-1}} e^{-x} - x$$

2. a) Use separation of variables, $u(r,\theta) = R(r)T(\theta)$, to find ordinary differential equations which $R(r)$ and $T(\theta)$ must satisfy if $u(r,\theta)$ is to be a solution of

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

Do not solve these equations.

Solution: Let $u(r,\theta) = R(r)T(\theta)$ then

$u_r = R'(r)T(\theta) \qquad u_{rr} = R''(r)T(\theta) \qquad u_{\theta\theta} = R(r)T''(\theta)$

and $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ becomes

$$R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) = 0$$

$$r^2R''(r)T(\theta) + rR'(r)T(\theta) = -R(r)T''(\theta)$$

$$\frac{r^2R''(r) + rR'(r)}{-R(r)} = \frac{T''(\theta)}{T(\theta)} = k$$

since R and T are independent
resulting in the equations

$$r^2R''(r) + rR'(r) + kR(r) = 0$$

and

$$T''(\theta) - kT(\theta) = 0$$

2. b)Consider the non-homogeneous problem

P.D.E.: $u_{xx} = 9u_t$

B.C.'s: $u_x(0,t) = 0 \qquad u(1,t) = 2$

I.C.: $u(x,0) = -3\cos\frac{7\pi}{2}x + 2$

i) (5 points)

Let $v(x,t) = u(x,t) - 2$ and show that $v(x,t)$ satisfies the homogeneous problem

P.D.E.: $v_{xx} = 9v_t$

B.C.: $v_x(0,t) = 0 \qquad v(1,t) = 0$

I.C.: $v(x,0) = -3\cos\frac{7\pi}{2}x$

Solution to i) $u_{xx}(x,t) = v_{xx}(x,t) \qquad u_x(x,t) = v_x(x,t)$

$u_{tt}(x,t) = v_{tt}(x,t) \qquad u_t(x,t) = v_t(x,t)$

$u(1,t) = 2$ and $u(x,t) - 2 = v(x,t) \Rightarrow v(1,t) = 0$

$u_x(0,t) = 0 \Rightarrow v_x(0,t) = 0$

$u(x,0) = -3\cos\frac{7\pi}{2} + 2$ and $u(x,t) - 2 = v(x,t) \Rightarrow v(x,0) = -3\cos\frac{7\pi}{2}$

2. b) ii) (10 points)

Solve the above problem for $v(x,t)$.

Solution to ii) Let $v(x,t) = X(x)T(t)$

then $X''T = 9XT' \Rightarrow \frac{X''}{X} = 9\frac{T'}{T} = k$

resulting in the ordinary differential equations:

$X'' - kX = 0$ and $T' - \frac{k}{9}T = 0$

Boundary Conditions become: $X'(0)T(t) = 0$ and $X(1)T(t) = 0$

$\Rightarrow X'(0) = 0$ and $X(1) = 1$

Solving the differential equation $X'' - kX = 0$ consider all values of k

$k < 0$ let $k = -u^2; \quad u > 0$

$X'' + u^2 X = 0$ has the solution: $X(x) = c_1 \cos ux + c_2 \sin ux$

and $X'(x) = -c_1 u \sin ux + c_2 u \cos ux$

B.C. $\Rightarrow X(1) = c_1 \cos u + c_2 \sin u = 0$ and $X'(0) = c_2 u = 0$

$\Rightarrow c_2 = 0$ thus $c_1 \cos u = 0 \Rightarrow u_n = \frac{(2n-1)\pi}{2} \quad n = 1, 2, \ldots$

$\Rightarrow k_n = -\frac{(2n-1)^2\pi^2}{4} \quad n = 1, 2, \ldots$

so $X_n(x) = c_n \cos \frac{(2n-1)\pi}{2} x$

$k = 0 \Rightarrow X'' = 0$ which has the solution: $X(x) = c_1 x + c_2$ and $X'(x) = c_1$

B.C. $\Rightarrow X(1) = c_1 + c_2 = 0$ and $X'(0) = c_1 = 0 \Rightarrow c_2 = 0$

thus $X(x) \equiv 0$ is the trivial solution.

$k > 0$ let $k = u^2; \quad u > 0$

$X'' - u^2 X = 0$ has the solution: $X(x) = c_1 e^{ux} + c_2 e^{-ux}$

and $X'(x) = c_1 u e^{ux} - c_2 u e^{-ux}$

B.C. $\Rightarrow X'(0) = c_1 u - c_2 u = 0 \Rightarrow c_1 = c_2$

and $X(1) = c_1 e^u + c_2 e^{-u} = 0 \Rightarrow c_1 e^u + c_1 e^{-u} = 0 \Rightarrow c_1(e^u + e^{-u}) = 0$

$\Rightarrow c_1 = c_2 = 0$ thus $X(x) \equiv 0$ is the trivial solution.

Using the non-trivial solution $k_n = -\frac{(2n-1)^2\pi^2}{4} \qquad X_n(x) = c_n \cos \frac{(2n-1)\pi}{2} x,$

the equation $T' - \frac{k}{9} T = 0$ becomes $T' + \frac{(2n-1)^2\pi^2}{36} T = 0$

solving by separating $\frac{T'}{T} = -\frac{(2n-1)^2\pi^2}{36} \Rightarrow \int \frac{T'}{T} = -\int \frac{(2n-1)^2\pi^2}{36}$

$\Rightarrow \ln T = -\frac{(2n-1)^2\pi^2}{36} t + c \Rightarrow T_n(t) = c_n e^{-\frac{(2n-1)^2\pi^2}{36} t}$

Therefore $v_n(x,t) = X_n(x) T_n(t)$

$v_n(x,t) = c_n \cos \frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2\pi^2}{36} t}$

so $v(x,t) = \sum_{n=1}^{\infty} c_n \cos \frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2\pi^2}{36} t}$

Using I.C. to compute coefficients:

$$v(x,0) = \sum_{n=1}^{\infty} c_n \cos \frac{(2n-1)\pi x}{2} = -3 \cos \frac{7\pi x}{2}$$

by equating coefficients: $c_1 = 0, c_2 = 0, c_3 = -3, c_4 = 0, \ldots$

$$v(x,t) = -3 \cos \frac{7\pi x}{2} e^{-\frac{49\pi^2}{36} t}$$

is the solution.

2. b)iii) (2 points)

Now use the results of b) i) and ii) to find $u(x,t)$.

Solution to iii)

$$u(x,t) = -3\cos\frac{7\pi x}{2}e^{-\frac{49\pi^2}{36}t} + 2$$

Ma 227 Final Exam 98S

Problem 1
a) (8 points)
Find the first four nonzero terms of the Fourier cosine series of

$$f(x) = \begin{cases} -1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

Solution
If $f(x)$ is a function defined on $[0, L]$, then its Fourier cosine expansion is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

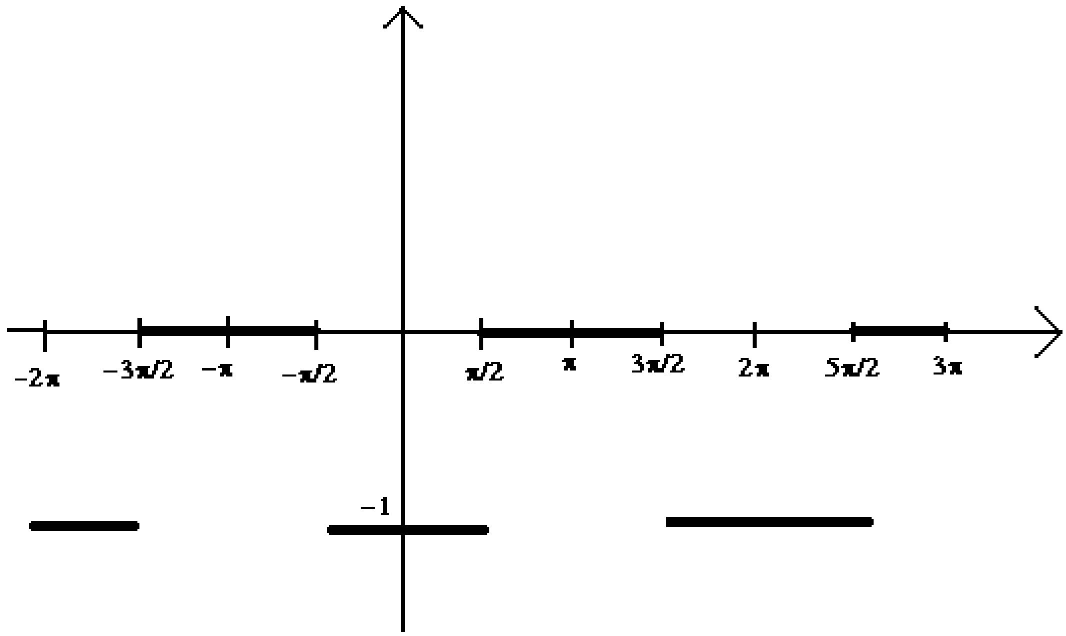
where $a_0 = \frac{1}{L}\int_0^L f(x)dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \qquad n = 1, 2, 3, \ldots$
Here $L = \pi$ so that $f(x) = \sum_{n=1}^{\infty} a_n \cos(nx), a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$.

Thus $a_0 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}(-1)dx + \frac{1}{\pi}\int_0^{\frac{\pi}{2}}(0)dx = -\frac{1}{2}$. Also,

$$a_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}(-1)\cos nxdx = -\frac{2}{n\pi}[\sin nx]_0^{\frac{\pi}{2}} = -\frac{2}{n\pi}\left[\sin\frac{n\pi}{2}\right]$$

Therefore
$a_1 = -\frac{2}{\pi}, \qquad a_2 = 0, \qquad a_3 = +\frac{2}{3\pi}, \qquad a_4 = 0, \qquad a_5 = -\frac{2}{5\pi}, \qquad a_6 = 0, \qquad a_7 = +\frac{2}{7\pi}$

Hence
$f(x) = -\frac{1}{2} - -\frac{2}{\pi}\cos x + 0 \bullet \cos 2x + \frac{2}{3\pi}\cos 3x + 0 \bullet \cos 4x - \frac{2}{5\pi}\cos 5x + 0 \bullet \cos 6x + \frac{2}{7\pi}\cos 7x$
b) (8 points)
Sketch the graph of the function to which the Fourier series in (a) converges
on $-2\pi < x < 3\pi$.



c) (9 points)
Find the eigenvalues and eigenfunctions for the problem

$y'' + \lambda y = 0$; $y(0) = 0$; $y(2) = 0$

Be sure to check the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.
I. Consider the case $\lambda < 0$ first. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. The DE becomes

$y'' - \alpha^2 y = 0$.

The general solution of this equation is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. Thus

$y(0) = c_1 + c_2 = 0$ and $y(2) = c_1 e^{2\alpha} + c_2 e^{-2\alpha} = 0$.

The first equation implies that $c_1 = -c_2$. Thus the second equation becomes $c_1(e^{2\alpha} + e^{-2\alpha}) = 0$. Thus $c_1 = 0$; this tells us that $c_2 = 0$ also. Therefore $y = 0$ is the only solution if $\lambda < 0$. Hence there are no negative eigenvalues.

II. Suppose $\lambda = 0$. The DE becomes $y'' = 0$ which has the solution $y = c_1 x + c_2$. The boundary conditions imply $y(0) = c_1 = 0$, so that $y = c_2$. But $y(2) = c_2 = 0$ so that $y = 0$. Hence there is no eigenfunction corresponding to the eigenvalue $\lambda = 0$.

III. Suppose $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE becomes

$y'' + \beta^2 y = 0.$

The general solution of this equation is $y(x) = c_1 \sin\beta x + c_2 \cos\beta x.$ Thus

Now $y(0) = c_2 = 0$ Thus $y(x) = c_2 \sin\beta x.$ Now $y(2) = c_2 \sin 2\beta = 0.$ For a nontrivial solution we must have $c_2 \neq 0.$ This means that $\sin 2\beta = 0$ or $\beta = \frac{n\pi}{2},\ n = 1, 2, 3, \ldots$ The eigenvalues are therefore $\lambda = \beta^2 = \frac{n^2\pi^2}{4}$ and the corresponding eigenfunctions are $y_n = a_n \sin\frac{n\pi}{2}x,\ n = 1, 2, 3, \ldots$

Problem 2

a) (10 points)

Use separation of variables, $u(x,t) = X(x)T(t)$, to find ordinary differential equations which $X(x)$ and $T(t)$ must satisfy if $u(x,t)$ is to be a solution of

$$5x^5t^2u_{tt} + (t+3)^5(x+5)^2u_{xx} = 0$$

Do not solve these equations.

Solution:

$u_x = X'T, \qquad u_{xx} = X''T, \qquad\qquad u_t = XT', \quad u_{tt} = XT''$

Thus the given equation becomes

$15t^2x^5XT'' + (t+3)^5(x+5)^2X''T = 0$

$$\Rightarrow \qquad 15x^5\frac{X}{(x+5)^2X''} = -(t+3)^5\frac{T}{t^2T''} = k, \qquad k \text{ a constant}$$

This yields the two ODEs

$15x^5X - k(x+5)^2X'' = 0$

$(t+3)^5T + kt^2T'' = 0$

b) (15 points)

Solve:

P.D.E.: $u_{xx} = 4u_t$

B.C.s: $u(0,t) = u(2,t) = 0$

I.C.: $u(x,0) = -3\sin\frac{\pi x}{2} + 23\sin\pi x - 4\sin 2\pi x$

Let $u(x,t) = X(x)T(t).$ Then differentiating and substituting in the PDE yields

$X''T = 4XT'$

$\Rightarrow \qquad \frac{X''}{X} = 4\frac{T'}{T}$

Using the argument that the left hand side is purely a function of x and the right hand side is purely a function of t, and the only way that they can be equal is if they are equal to a constant, we get

$$\frac{X''}{X} = 4\frac{T'}{T} = k \qquad k \text{ a constant}$$

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T' - \frac{1}{4}kT = 0$$

The boundary condition $u(0,t) = 0$ implies that $X(0)T(t) = 0$. We cannot have $T(t) = 0$, since this would imply that $u(x,t) = 0$. Thus $X(0) = 0$. Similarly, the boundary condition $u(2,t) = 0$ leads to $X(2) = 0$.

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X(0) = X(2) = 0$$

This boundary value problem is the one given in Problem 1(c) above with $k = -\lambda$. The solution is

$$k = -\left(\frac{n\pi}{2}\right)^2 \qquad X_n(x) = a_n \sin\frac{n\pi}{2}x \qquad n = 1,2,3,\ldots$$

Substituting the values of k into the equation for $T(t)$ leads to

$$T' + \frac{n^2\pi^2}{16}T = 0$$

which has the solution $T_n(t) = c_n e^{-\frac{n^2\pi^2 t}{16}}$, $n = 1,2,3,\ldots$

We now have the solutions

$$u_n(x,t) = A_n \sin\frac{n\pi}{2}x e^{-\frac{n^2\pi^2 t}{16}} \qquad n = 1,2,3,\ldots$$

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{2} x e^{-\frac{n^2\pi^2 t}{16}}$$

satisfies the PDE and the boundary conditions. Since

$$u(x,0) = -3\sin\frac{\pi x}{2} + 23\sin\pi x - 4\sin 2\pi x = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{2} x.$$

Matching the cosine terms on both sides of this equation leads to

$A_1 = -3 \qquad A_2 = 23$ and $A_4 = -4.$ All of the other constants must be zero, since there are no sine terms on the left to match with them. Thus

$$u(x,t) = -3\sin\frac{\pi x}{2} e^{-\frac{\pi^2}{16}t} + 23\sin\pi x e^{-\frac{\pi^2}{4}t} - 4\sin 2\pi x e^{-\pi^2 t}$$