Ma 221 Homework Solutions Due Date: January 26, 2010

2.2 pg. 46 # 1, 4, 5, 7, 9, 15, 19, 21, 23; 2.3 p.54-55 #1, 3, 5, 7, 10, 17, 19, 21
(Underlined problems are handed in)

In problems 1, 4 and 5, determine whether the given differential equation is separable.

1) \( \frac{dy}{dx} = 4y^2 - 3y + 1 \Rightarrow \frac{dy}{4y^2 - 3y + 1} = dx \)
   therefore, this equation is separable.

4.) \( \frac{ds}{dt} = t \ln(s^2) + 8t^2 \Rightarrow \frac{ds}{dt} = 2t^2 \ln s + 8t^2 \Rightarrow \frac{ds}{2t^2 \ln s + 4} = 2t^2 dt \)
   therefore, this equation is separable.

5.) \( s^2 + \frac{ds}{dt} = \frac{s+1}{st} \)
   Writing the equation in the form
   \( \frac{ds}{dt} = \frac{s+1}{st} - s^2 \)
   shows that the equation is not separable.

In problems 7, 9, 11 and 15, solve the equation.

7) \( \frac{dy}{dx} = y(2 + \sin x) \)
   \( \frac{dy}{dx} = y(2 + \sin x) \Rightarrow \frac{dy}{y} = (2 + \sin x)dx \Rightarrow \int \frac{dy}{y} = \int (2 + \sin x)dx \Rightarrow \ln y = 2x - \cos x + C \)

9.) \( \frac{dy}{dx} = \frac{1-x^2}{y^2} \)
   \( \frac{dy}{dx} = \frac{1-x^2}{y^2} \Rightarrow y^2 dy = (1-x^2)dx \Rightarrow \int y^2 dy = \int (1-x^2)dx \Rightarrow \frac{1}{3} y^3 = x \)

11) \( \frac{dy}{dx} = \sec^2 y \frac{1+x^2}{1+x^2} \)
   \( \frac{dy}{dx} = \sec^2 y \frac{1+x^2}{1+x^2} \Rightarrow \frac{dy}{\sec^2 y} = \frac{dx}{1+x^2} \)

Using trigonometric identities we have:
\[
\Rightarrow \sec y = \frac{1}{\cos y} \quad \text{and} \quad \cos^2 y = \frac{1}{2}(1 + \cos 2y) \\
\Rightarrow \frac{dy}{\sec^2 y} = \frac{dx}{1+x^2} \quad \Rightarrow \quad \int \frac{(1+\cos 2y)dy}{2} = \int \frac{dx}{1+x^2} \\
\Rightarrow \quad \frac{1}{2}(y + \frac{1}{2}\sin 2y) = \arctan x + C_1 \Rightarrow \quad 2y + \sin 2y = 4\arctan x + 4C_1 \\
\Rightarrow \quad 2y + \sin 2y = 4\arctan x + C
\]

15.) \[y^{-1}dy + ye^{\cos x} \sin x dx = 0\]

\[-ye^{\cos x} \sin x dx = y^{-1}dy\]

\[-\int e^{\cos x} \sin x dx = \int y^{-2}dy\]

Substituting:

let \[u = \cos x\]
\[du = -\sin x dx\]
\[-du = \sin x dx\]

\[\int e^u du = \int y^{-2}dy\]

\[e^u = -y^{-1} + C\]
\[e^{\cos x} = y^{-1} + C\]
\[y = \frac{1}{C - e^{\cos x}}\]

In problems 17, 19, 21, 22, 23, and 25, solve the initial value problem.

17) \[y' = x^3(1 - y), \quad y(0) = 3\]

\[\frac{dy}{dx} = x^3(1 - y) \quad \Rightarrow \quad \frac{dy}{(1 - y)} = x^3 dx \quad \Rightarrow \quad \int \frac{dy}{(1 - y)} = \int x^3 dx \quad \Rightarrow \quad -\ln|1 - y| +\]

\[|1 - y| = \exp(C_1 - \frac{x^4}{4}) = Ce^{-\frac{x^4}{4}}\]

Substituting the IC \[y(0) = 3\]

\[|1 - 3| = Ce^{-\frac{0^4}{4}} \quad \Rightarrow \quad |-2| = C = 2\]

\[\Rightarrow |1 - y| = 2e^{-\frac{x^4}{4}}\]

Since \[1 - y(0) = 1 - 3 < 0\], on an interval containing \(x = 0\) one has \(1 - y(x) < 0\) and so \(|1 - y(x)| = y(x) - 1\). The solution is then:
19) \( \frac{dy}{dx} = 2\sqrt{y + 1} \cos x, \quad y(\pi) = 0 \)

\[ \frac{dy}{2\sqrt{y + 1}} = \cos x \, dx = \frac{1}{2}(y + 1)^{-1/2} \, dy \]

\[ \Rightarrow \frac{1}{2}(y + 1)^{-1/2} \, dy = \cos x \, dx \]

\[ \int \frac{1}{2}(y + 1)^{-1/2} \, dy = \int \cos x \, dx \]

\( (y + 1)^{1/2} = \sin x + C \)

Substituting the IC

\( y(\pi) = 0 \Rightarrow (0 + 1)^{1/2} = \sin \pi + C \Rightarrow C = 1 \)

\( (y + 1)^{1/2} = \sin x + 1 \Rightarrow y = (\sin x + 1)^2 - 1 = \sin^2 x + 2 \sin x \)

21.) \( \frac{dy}{d\theta} = y \sin \theta, \quad y(\pi) = -3 \)

\[ \frac{dy}{d\theta} = y \sin \theta \Rightarrow \frac{dy}{y} = \sin \theta \, d\theta \Rightarrow \int \frac{dy}{y} = \int \sin \theta \, d\theta \Rightarrow \ln |y| = -\cos \theta + C_1 \]

because at the initial point, \( \theta = \pi, \quad y(\pi) = -3 < 0 \)

\[ -3 = y(\pi) = -Ce^{-\cos \pi} = -Ce^1 \Rightarrow C = 3e^{-1} \Rightarrow y = -3e^{-1}e^{-\cos \theta} = -3e^{-1-\cos \theta} \]

22) \( x^2 \, dx + 2y \, dy = 0, \quad y(0) = 2 \)

\[ x^2 \, dx = -2y \, dy \Rightarrow \int x^2 \, dx = \int -2y \, dy \Rightarrow \frac{x^3}{3} = -y^2 + C \]

Substituting the IC \( y(0) = 2 \):

\[ \frac{0^3}{3} = -2^2 + C \Rightarrow C = 4 \]

\[ \frac{x^3}{3} = -y^2 + 4 \Rightarrow y = \sqrt{4 - \frac{x^3}{3}} \]

23.) \( \frac{dy}{dt} = 2t \cos^2 y, \quad y(0) = \frac{\pi}{4} \)

\[ \frac{dy}{\cos^2 y} = 2t \, dt \]

\[ \int \sec^2 y \, dy = \int 2t \, dt \]

\[ \tan y = t^2 + C \]

\[ \tan \left( \frac{\pi}{4} \right) = C \]

\[ C = 1 \]

Thus

\[ \tan y = t^2 + 1 \]

or

\[ y = \arctan(t^2 + 1) \]
25.) \( \frac{dy}{dx} = x^2(1 + y) \quad y(0) = 3 \)

\[
\frac{dy}{dx} = x^2(1 + y) \Rightarrow \int \frac{dy}{1+y} = \int x^2 \, dx \Rightarrow \ln|1 + y| = \frac{1}{3}x^3 + C_1 \Rightarrow 1 + y = e^{\frac{1}{3}x^3 + C_1}
\]

\[
y = Ce^{\frac{1}{3}x^3} - 1 \quad 3 = y(0) = C - 1 \Rightarrow C = 4 \Rightarrow y = 4e^{\frac{1}{3}x^3} - 1
\]

2.3 p. 54 to p. 55 #1, 3, 5, 7, 10, 11, 13, 15, 17, 18, 19, 21, 22, 30

(Underlined Problems are handed in)

For problems 1, 3, and 5 determine whether the given equation is separable, linear, neither, or both.

1.) \( x^2 \frac{dy}{dx} + \sin x = y \)

Isolating \( \frac{dy}{dx} \) we get:

\[
\frac{dy}{dx} = \frac{y - \sin x}{x^2}. \quad \text{Since the right hand side cannot be represented as a product } g(x)p(y), \text{ the equation is not separable.}
\]

3.) \( (t^2 + 1) \frac{dy}{dt} = yt - y \)

This is a linear equation with independent variable \( t \) and dependent variable \( y \). This is a separable equation as shown:

\[
\frac{dy}{dt} = \frac{y(t-1)}{(t^2+1)} \Rightarrow \frac{dy}{y} = \frac{(t-1)}{(t^2+1)} \, dt \Rightarrow \frac{dy}{dt} = \frac{(t-1)}{(t^2+1)} y = g(t)p(y)
\]

5.) \( x \frac{dx}{dt} + t^2x = \sin t \)

In this equation, the independent variable is \( t \) and the dependent variable is \( x \). Dividing by \( x \) we obtain

\[
\frac{dx}{dt} = \frac{\sin t}{x} - t^2.
\]

Therefore, it is neither linear, because of the \( \frac{\sin t}{x} \) term, nor separable, because the righthand side is not a product of functions of single variables \( x \) and \( t \).

For problems 7, 10, 11, 13 and 15, obtain the general solution to the equation

7.) \( \frac{dy}{dx} - y = e^{3x} \)

In this equation, \( P(x) = -1 \) and \( Q(x) = e^{3x} \)

The integrating factor is thus: \( \mu(x) = \exp \left( \int P(x) \, dx \right) = \exp \left( \int (-1) \, dx \right) = e^{-x} \)

Multiplying both sides of the equation by \( \mu(x) \) and integrating yields:

\[
e^{-x} \frac{dy}{dx} - e^{-x}y = e^{-x}e^{3x} = e^{2x} \Rightarrow \frac{d(e^{-x}y)}{dx} = e^{2x}
\]

\[
\Rightarrow e^{-x}y = \int e^{2x} \, dx = \frac{1}{2}e^{2x} + C \Rightarrow y = \left( \frac{1}{2}e^{2x} + C \right)e^x = \frac{e^{3x}}{2} + Ce^x
\]
10.) \( x \frac{dy}{dx} + 2y = x^{-3} \)

Putting the equation in standard form:
\[ \frac{dy}{dx} + \frac{2}{x}y = x^{-4} \]

Find \( \mu(x) = e^{\int P \, dx} = e^{\int \left( \frac{2}{x} \right) \, dx} = e^{2 \ln|x|} = (|x|)^2 = x^2 \)

Multiply through by \( \mu(x) \) to get
\[ x^2 \frac{dy}{dx} + 2xy = x^{-2} \]

Integrate to get
\[ \int x^2 \, dy = \int x^{-2} \, dx \]
\[ yx^2 = -x^{-1} + C \]

Solve explicitly for \( y \)
\[ \Rightarrow y = -x^{-3} + Cx^{-2} \]

11.) \( (t + y + 1) \, dt - dy = 0 \)

Choosing \( t \) as the independent variable and \( y \) as the dependent variable, the equation can be put into standard form:
\[ t + y + 1 - \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} - y = t + 1 \]

Thus: \( P(t) = -1 \) and \( \mu(t) = \exp \left[ \int (-1) \, dt \right] = e^{-t} \)

Multiplying both sides by \( \mu(t) \) and integrating yields:
\[ e^{-t} \frac{dy}{dt} - e^{-t}y = (t + 1)e^{-t} \Rightarrow \frac{d(e^{-t}y)}{dt} = (t + 1)e^{-t} \]
\[ \Rightarrow e^{-t}y = \int (t + 1)e^{-t} \, dt = -(t + 1)e^{-t} + \int e^{-t} \, dt = -(t + 1)e^{-t} - e^{-t} + C = -(t + 2)e^{-t} + C \]
\[ \Rightarrow y = e^{t}(-(t + 2)e^{-t} + C) = -t - 2 + Ce^{t} \]

13.) \( y \frac{dx}{dy} + 2x = 5y^{3} \)

In this problem, the independent variable is \( y \) and the dependent variable is \( x \). So, we divide the equation by \( y \) to rewrite it in standard form.
\[ \frac{dx}{dy} + \frac{2}{y}x = 5y^{2} \Rightarrow \frac{dx}{dy} + \frac{2}{y}x = 5y^{2} \]

Therefore, \( P(y) = \frac{2}{y} \) and the integrating factor, \( \mu(y) \), is
\[ \mu(y) = e^{\int \frac{2}{y} \, dy} = e^{2 \ln|y|} = |y|^2 = y^2 \]

Multiplying the equation (in standard form) by \( y^2 \) and integrating yields
\[ \left( \frac{dx}{dy} + \frac{2}{y}x = 5y^{2} \right)y^2 = y^2 \frac{dx}{dy} + 2yx = 5y^{4} \Rightarrow \frac{d}{dy} (y^{2}x) = 5y^{4} \]
\[ \Rightarrow y^2x = \int 5y^{4} \, dy = y^{5} + C \]
\[ \Rightarrow x = y^{-2}(y^{5} + C) = y^{3} + Cy^{-2} \]

15.) \( (x^2 + 1) \frac{dy}{dx} + xy - x = 0 \)

Divide by \( (x^2 + 1) \)
\[ \frac{dy}{dx} + \frac{x}{x^2 + 1} y = \frac{x}{x^2 + 1} \]

so \( P(x) = \frac{x}{x^2 + 1} \)

Find \( \mu(x) = e^{\int P(x) dx} \)

\[ \mu(x) = e^{\int \frac{x}{x^2 + 1} dx} = e^{\frac{1}{2} \ln(x^2 + 1)} = (x^2 + 1)^{1/2} \]

Multiply through by \( \mu(x) \) to get

\[ (x^2 + 1)^{1/2} y' + \frac{x}{(x^2 + 1)^{1/2}} y = \frac{x}{(x^2 + 1)^{1/2}} \text{ or} \]

\[ \frac{d}{dx} \left( (x^2 + 1)^{1/2} y \right) = \frac{x}{(x^2 + 1)^{1/2}} \]

Integrating gives

\[ y(x^2 + 1)^{1/2} = (x^2 + 1)^{1/2} + C \]

Solve explicitly for \( y \)

\[ y = 1 + C(x^2 + 1)^{-1/2} \]

For problems 17, 18 and 19, 21, 22 solve the initial value problems.

17.)

\[ \frac{dy}{dx} = \frac{y}{x} = xe^x \quad y(1) = e - 1 \]

This is a linear equation with \( P(x) = -1/x \) and \( Q(x) = xe^x \). The integrating factor is given by:

\[ \mu(x) = e^{\int P(x) dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}; \quad \text{For } x > 0 \]

Multiply through by \( \mu(x) \) to get

\[ \frac{1}{x} y' - \frac{y}{x^2} = e^x \]

or

\[ \frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = e^x \]

Integrate to get

\[ \frac{y}{x} = e^x + C \]

Solve explicitly for \( y \)

\[ y = xe^x + Cx \]

Plug in initial condition \( y(1) = e - 1 \) and solve for \( C \)

\[ e - 1 = e + C \Rightarrow C = -1 \]

Plug in the value for \( C \)

\[ y = xe^x - x \]

Using SNB to check our answer we have

\[ \frac{dy}{dx} = e^x, \text{ Exact solution is: } xe^x - x \]

\[ y(1) = e - 1 \]

18.) \( \frac{dy}{dx} + 4y - e^{-x} = 0 ; y(0) = \frac{4}{5} \)

\[ \frac{dy}{dx} + 4y = e^{-x} \]

Find \( \mu(x) = e^{\int P(x) dx} = e^{\int 4 dx} = e^{4x} \)
Multiply through by \( \mu(x) \) to get
\[ e^{4x}y' + 4e^{4x}y = e^{3x} \]
or
\[ \frac{d}{dx} (e^{4x}y) = e^{3x} \]
Integrate to get
\[ ye^{4x} = \frac{1}{3}e^{3x} + C \]
Solve explicitly for \( y \)
\[ y = \frac{1}{3}e^{-x} + \frac{C}{e^{4x}} \]
Plug in initial condition \( y(0) = \frac{4}{3} \) and solve for \( C \)
\[ \frac{4}{3} = \frac{1}{3}e^{0} + \frac{C}{e^{4(0)}} \]
\[ \frac{4}{3} = \frac{1}{3} + C \]
So \( C = 1 \)
Plug in the value for \( C \)
\[ y = \frac{1}{3}e^{-x} + e^{-4x} \]
Using SNB to check our answer we have
\[ \frac{dy}{dx} + 4y - e^{-x} = 0 \]
, Exact solution is: \( y(x) = \frac{1}{3}e^{-x} + e^{-4x} \)

19.) \( t^3 \frac{dx}{dt} + 3t^2x = t, \quad x(2) = 0 \)
In this problem, \( t \) is the dependent variable and \( x \) is the dependent variable. Notice the left side of the equation is the derivative of \( xt^3 \) with respect to \( t \). Using the product rule for differentiation yields:
\[ \frac{d}{dt} (xt^3) = \frac{dx}{dt} t^3 + x \frac{d(t^3)}{dt} = t^3 \frac{dx}{dt} + 3t^2x \]
The equation becomes:
\[ \frac{d}{dt} (xt^3) = t \quad \Rightarrow \quad xt^3 = \int t dt = \frac{t^2}{2} + C \]
\[ \Rightarrow x = t^3 \left( \frac{t^2}{2} + C \right) = \frac{1}{2} + \frac{C}{t^3} \quad \Rightarrow \quad x = \frac{1}{2} + \frac{C}{t^3} \]
\[ 0 = x(2) = \frac{1}{2(2)} + \frac{C}{2^3} \quad \Rightarrow \quad \frac{1}{4} + \frac{C}{8} = 0 \quad \Rightarrow \quad C = -2 \quad \Rightarrow \quad x = \frac{1}{2} - \frac{2}{t^3} \]

21.) \( \cos x \frac{dy}{dx} + y \sin x = 2x \cos^2 x, \quad y(\frac{\pi}{4}) = \frac{-15\sqrt{2} \pi^2}{32} \)
Putting the equation in standard form:
\[ \frac{dy}{dx} + \frac{\sin x}{\cos x} y = 2x \cos^2 x \quad \Rightarrow \quad \frac{dy}{dx} + (\tan x)y = 2x \cos x \]
Find \( \mu(x) = e^{\int \tan x dx} = e^{(-\ln|\cos x|)} = |\cos x|^{-1} \)
At the initial point, \( x = \frac{\pi}{4}, \cos \frac{\pi}{4} > 0 \) therefore we can take \( \mu(x) = (\cos x)^{-1} \)
Multiply through by \( \mu(x) \) to get
\[ \frac{1}{\cos x} \frac{dy}{dx} + \frac{\sin x}{\cos^2 x} y = 2x \quad \Rightarrow \quad \frac{d}{dx} \left( \frac{y}{\cos x} \right) = 2x \]
Integrate to get
\[ \int \frac{d}{dx} \left( \frac{y}{\cos x} \right) = \int 2x \]
\[ \frac{y}{\cos x} = x^2 + C \]
Solve explicitly for \( y \)
\[ y = x^2 \cos x + C \cos x \]

Plug in initial condition \[ y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2} \pi^2}{32} \] and solve for \( C \)

\[ \frac{-15\sqrt{2} \pi^2}{32} = \frac{\pi^2}{4} \cos \frac{\pi}{4} + C \cos \frac{\pi}{4} \]

So \( C = -\pi^2 \)

Plug in the value for \( C \)
\[ y = x^2 \cos x - \pi^2 \cos x = \cos x(x^2 - \pi^2) \]

22.) \( \sin x \frac{dy}{dx} + y \cos x = x \sin x, \quad y\left(\frac{\pi}{2}\right) = 2 \)

Putting the equation in standard form:

\[ \frac{dy}{dx} + \frac{\cos x}{\sin x} y = x \Rightarrow \frac{dy}{dx} + (\cot x)y = x \]

Find \( \mu(x) = e^{\int p dx} = e^{\int \frac{\cos x}{\sin x} dx} = e^{\ln(\sin x)} = \sin x \)

Multiply through by \( \mu(x) \) to get

\[ \sin x \frac{dy}{dx} + y \cos x = x \sin x \Rightarrow \frac{d}{dx}(y \sin x) = x \sin x \]

Integrate to get

\[ \int \frac{d}{dx}(y \sin x) = \int x \sin x \quad \text{(Using integration by parts)} \]

\[ y \sin x = \sin x - x \cos x + C \]

Solve explicitly for \( y \)

\[ y = 1 - x \cos x \sin x + \frac{C}{\sin x} = 1 - x \cot x + \frac{C}{\sin x} \]

Plug in initial condition \( y\left(\frac{\pi}{2}\right) = 2 \) and solve for \( C \)

\[ 2 = 1 - \frac{\pi}{2} \cot \frac{\pi}{2} + \frac{C}{\sin \frac{\pi}{2}} \]

So \( C = 1 \)

Plug in the value for \( C \)
\[ y = 1 - x \cot x + \frac{1}{\sin x} = 1 - x \cot x + \csc x \]

30.) Show that the substitution \( v = y^3 \) reduces equation \( \frac{dy}{dx} + 2y = xy^{-2} \) to the equation \( \frac{dv}{dx} + 6v = 3x \). Then solve the equation for \( v \), and make the substitution \( v = y^3 \) to obtain the solution the equation \( \frac{dy}{dx} + 2y = xy^{-2} \).

\[ \frac{dv}{dx} + 6v = 3x \]

Divide by \( y^{-2} \)

\[ y^2 \frac{dv}{dx} + 2y^3 = x \]

\[ v = y^3 \]

Differentiate \( v \) with respect to \( x \)
\[ \frac{dv}{dx} = 3y^2 \frac{dy}{dx} \]

Divide by 3

\[ \frac{1}{3} \frac{dv}{dx} = y^2 \frac{dy}{dx} \]

Notice that \( \frac{1}{3} \frac{dv}{dx} \) is equal to the first term on the left hand side of the equation. Make that substitution.

\[ \frac{1}{3} \frac{dv}{dx} + 2v = x \]

Now multiply by 3 to get a first order linear differential equation.
\[ \frac{dv}{dx} + 6v = 3x \]
Find \( \mu(x) \)

\[
\mu(x) = e^{\int 6 \, dx} = e^{6x}
\]

Multiply through by \( \mu(x) \) to get

\[
e^{6x} \frac{dv}{dx} + 6e^{6x} v = \frac{d}{dx} (e^{6x} v) = 3xe^{6x}
\]

Now integrate

\[
ve^{6x} = \int e^{6x} (3x) \, dx = \frac{1}{2} xe^{6x} - \frac{1}{12} e^{6x} + C
\]

Where SNB was used to evaluate the integral.

Solve explicitly for \( v \) yields

\[
v = \frac{1}{2} x - \frac{1}{12} + Ce^{-6x}
\]

Plug \( y \) back into the equation from the substitution \( v = y^3 \) and solve for \( y \).

\[
y^3 = \frac{1}{2} x - \frac{1}{12} + Ce^{-6x}
\]

or

\[
y = \left( \frac{1}{2} x - \frac{1}{12} + Ce^{-6x} \right)^{\frac{1}{3}}
\]