MA 221 Homework Solutions
Due date: February 23, 2010

Section 4.5 pg. 201 1b, 2b, 5, 7, 17, 21, 23, 27, 29
(Underlined Problems are to be handed in)

1b) Given that $y_1(t) = \cos t$ is a solution to $y'' - y' + y = \sin t$ and that $y_2(t) = e^{2t}/3$ is a solution to $y'' - y' + y = e^{3t}$, use the superposition principle to find a solution of

$$y'' - y' + y = \sin t - 3e^{2t}$$

$y_p = \cos t - e^{2t}$

2.) Given that $y_1(t) = (1/4) \sin 2t$ is a solution to $y'' + 2y' + 4y = \cos 2t$ and that $y_2(t) = t/4 - 1/8$ is a solution to $y'' + 2y' + 4y = t$, use the superposition principle to find solutions of

(b) $y'' + 2y' + 4y = 2t - 3 \cos 2t$

Since the right hand side of this last equation is twice the right hand side of the second equation given plus $-3$ times the right hand side of the first equation given a particular solution for (b) is

$$2y_2 - 3y_1 = \frac{t}{2} - \frac{1}{4} - \frac{3}{4} \sin 2t$$

In problems 5 & 7 a nonhomogeneous equation and a particular solution are given. Find a general solution for the equation.

5.) $\theta'' - \theta' - 2\theta = 1 - 2t \quad \theta_p(t) = t - 1$

The homogeneous equation is

$$\theta'' - \theta' - 2\theta = 0$$

Thus $r^2 - r - 2 = (r-2)(r+1) = 0$ so $r = 2, -1$

$y_h = c_1 e^{2t} + c_2 e^{-t}$

Therefore by the superposition principle:

$y = \theta_h + \theta_p = t - 1 + c_1 e^{2t} + c_2 e^{-t}$

7.) $y'' - 2y' + 2e^x \quad y_p(x) = x^2 e^x$

The homogeneous equation is

$$y'' - 2y' + y = 0$$

Thus $r^2 - 2r + 1 = (r-1)^2 = 0$ so $r = 1$ is a repeated root and

$y_h = c_1 e^x + c_2 xe^x$

Therefore

$y = y_h + y_p = c_1 e^x + c_2 xe^x + x^2 e^x$

17.) $y'' - y = -11t + 1$

The indicial equation is $r^2 - 1 = 0$ so $r = \pm 1$ and

$$y_h = c_1 e^t + c_2 e^{-t}$$

Let
Therefore

\[ y_p = At + B \]
\[ y'_p = A \]
\[ y''_p = 0 \]

The DE implies

\[-At - B = -11t + 1\]

so \( A = 11, B = -1 \) and

\[ y_p = 11t - 1 \]

Thus

\[ y = y_h + y_p = c_1 e^t + c_2 e^{-t} + 11t - 1 \]

In problem 21, find a general solution to the differential equation.

21.) \( y''(\theta) + 2y'(\theta) + 2y(\theta) = e^{-\theta} \cos \theta \)

Let \( p(r) = r^2 + 2r + 2 = 0 \)

Then \( r = -1 \pm i \)

\[ y_h(\theta) = c_1 e^{-\theta} \cos \theta + c_2 e^{-\theta} \sin \theta = (c_1 \cos \theta + c_2 \sin \theta)e^{-\theta} \]

We now find a particular solution.

Method 1:

\[ y_p(\theta) = \theta(A \cos \theta + B \sin \theta)e^{-\theta} \]
\[ y'_p(\theta) = (A \cos \theta + B \sin \theta)e^{-\theta} + \theta[(A \cos \theta + B \sin \theta)e^{-\theta}]' \]
\[ y''_p(\theta) = 2[(A \cos \theta + B \sin \theta)e^{-\theta}]' + \theta[(A \cos \theta + B \sin \theta)e^{-\theta}]'' = 2[(B - A) \cos \theta - (B + A) \sin \theta] \]
\[ y''_p + 2y'_p + 2y_p = 2[(B - A) \cos \theta - (B + A) \sin \theta]e^{-\theta} + 2(A \cos \theta + B \sin \theta)e^{-\theta} = 2(B \cos \theta - A \sin \theta) \]

\( A = 0, B = 1/2, \) so \( y_p(\theta) = (1/2) \theta e^{-\theta} \sin \theta \)

Method 2: Consider a companion \( y''(\theta) + 2y'(\theta) + 2y(\theta) = e^{-\theta} \sin \theta. \) Multiply this equation by \( i \) and add it to the original equation. Letting \( w = y + iv, \) we have

\[ w'' + 2w' + 2w = e^{-\theta}(\cos \theta + i \sin \theta) = e^{-\theta} e^{i\theta} = e^{(i-1)\theta} \]

Then \( p(i - 1) = (i - 1)^2 + 2(i - 1) + 2 = -2i + 2i - 1 + 2 = 0. \) \( p'(r) = 2r + 2 \) so \( p(i - 1) = 2(i - 1) + 2 = 2i \)

Therefore

\[ w_p = -\frac{\theta e^{(i-1)\theta}}{2} = \frac{\theta e^{-\theta}(\cos \theta + i \sin \theta)}{2i} = \frac{-1}{2} \theta e^{-\theta}(i \cos \theta - \sin \theta) \]

Thus

\[ y_p = \text{Re} w_p = \frac{1}{2} \theta e^{-\theta} \sin \theta \]

Therefore

\[ y = y_h + y_p = c_1 e^{-\theta} \cos \theta + c_2 e^{-\theta} \sin \theta + \frac{1}{2} \theta e^{-\theta} \sin \theta \]

23.) \( y' - y = 1, \ y(0) = 0 \)
This is a first order linear equation. Multiply by \( e^{\int -dx} = e^{-x} \). The DE becomes
\[
\frac{d}{dx} (e^{-x}y) = e^{-x}
\]
so
\[
e^{-x}y = -e^{-x} + c
\]
or
\[
y = -1 + ce^x
\]
Then
\[
y(0) = -1 + c = 0
\]
and \( c = 1 \)
\[
y = -1 + e^x
\]

In problems 27 and 29, find the solution to the Initial Value Problem.

27. \( y''(x) - y'(x) - 2y(x) = \cos x - \sin 2x; \quad y(0) = -7/20, \quad y'(0) = 1/5 \)
\( p(r) = r^2 - r - 2 = 0 = (r - 2)(r + 1) \Rightarrow r = -1; r = 2 \)
\( y_h(x) = c_1 e^{-x} + c_2 e^{2x} \)

Method 1:
To find a particular solution for \( \cos x \) we consider the two equations
\[
y'' - y' - 2y = \cos x
\]
\[
v'' - v' - 2v = \sin x
\]
Multiply the second equation by \( i \) and add it to the first equation and let \( w = y + iv \) to get
\[
w'' - w' - 2w = \cos \theta + i \sin \theta = e^{i\theta}
\]
Since \( p(i) = -1 - i - 2 = -3 - i \neq 0 \) we have that
\[
w_p = \frac{ke^{i\theta}}{p(a)} = \frac{e^{i\theta}}{-3 - i} = -\frac{1}{3 + i} \left(\frac{3 - i}{3 - i}\right) (\cos \theta + i \sin \theta)
\]
\[
= \frac{3 - i}{10} (\cos \theta + i \sin \theta)
\]
Therefore \( y_{p1} = \text{real part of } w_p = -(3/10) \cos x - (1/10) \sin x \)

To find a particular solution for \( -\sin 2x \) we consider the two equations
\[
y'' - y' - 2y = -\sin 2x
\]
\[
v'' - v' - 2v = -\cos 2x
\]
Multiply the second equation by \( i \) and add it to the first equation and let \( w = v + iy \) to get
\[
w'' - w' - 2w = -\cos 2x - i \sin 2x = -e^{2i\theta}
\]
Since \( p(2i) = -4 - 2i - 2 = -6 - 2i \neq 0 \) we have that
\[
w_p = \frac{ke^{2i\theta}}{p(a)} = \frac{-e^{2i\theta}}{-6 - 2i} = \frac{1}{6 + 2i} \left(\frac{6 - 2i}{6 - 2i}\right) (\cos 2x + i \sin 2x)
\]
\[
= \frac{2(3 - i)}{40} (\cos 2x + i \sin 2x)
\]
Therefore \( y_{p2} = \text{imaginary part of } w_p = -(1/20) \cos 2x + (3/20) \sin 2x \)
\[
y = y_h + y_{p1} + y_{p2} = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{10} \cos x - \frac{1}{10} \sin x - \frac{1}{20} \cos 2x + \frac{3}{20} \sin 2x
\]
Method 2:
For the equation $y'' - y' - 2y = \cos x$

$y_{p,1}(x) = A \cos x + B \sin x$

$y'' - y' - 2y = (-A \cos x - B \sin x) - (-A \sin x + B \cos x) - 2(A \cos x + B \sin x)$

$= (-3A - B) \cos x + (A - 3B) \sin x = \cos x$

$A = -3/10; B = -1/10$

So, $y_{p,1}(x) = -(3/10) \cos x - (1/10) \sin x$

For the equation $y'' - y' - 2y = -\sin 2x$

$y_{p,2}(x) = A \cos 2x + B \sin 2x$

$y'' - y' - 2y = (-4A \cos 2x - 4B \sin 2x) - (-2A \sin 2x + 2B \cos 2x) - 2(A \cos 2x + B \sin 2x)$

$= (-6A - 2B) \cos 2x + (2A - 6B) \sin 2x = -\sin 2x$

$A = -1/20; B = +3/20$

So, $y_{p,2}(x) = -(1/20) \cos 2x + (3/20) \sin 2x$

so again we have

$$y = y_h + y_{p,1} + y_{p,2} = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{10} \cos x - \frac{1}{10} \sin x - \frac{1}{20} \cos 2x + \frac{3}{20} \sin 2x$$

$$y' = -c_1 e^{-x} + 2c_2 e^{2x} + \frac{3}{10} \sin x - \frac{1}{10} \cos x + \frac{1}{10} \sin 2x + \frac{3}{10} \cos 2x$$

Next, we find $c_1$ and $c_2$ such that the initial conditions are satisfied.

$$y(0) = c_1 + c_2 - \frac{3}{10} - \frac{1}{20} = -\frac{7}{20}$$

$$y'(0) = -c_1 + 2c_2 - \frac{1}{10} + \frac{3}{10} = \frac{1}{5}$$

Thus

$$c_1 + c_2 = 0$$

$$-c_1 + 2c_2 = 0$$

Hence $c_1 = c_2 = 0$

With these constants the solution is,

$$y(x) = -\frac{3}{10} \cos x - \frac{1}{10} \sin x - \frac{1}{20} \cos 2x + \frac{3}{20} \sin 2x$$

29.) $y''(\theta) - y(\theta) = \sin \theta - e^{2\theta}$ $y(0) = 1, \ y'(0) = -1$

The characteristic equation is $p(r) = r^2 - 1 = 0$, so $r = \pm 1$ and $y_h = c_1 e^{\theta} + c_2 e^{-\theta}$.

Method 1:
To find a particular solution for $\sin \theta$ we consider the two equations

$v'' - v = \cos \theta$

$y'' - y = \sin \theta$

Multiply the second equation by $i$ and add it to the first equation and let $w = v + iy$ to get

$w'' - w = \cos \theta + i \sin \theta = e^{i\theta}$
Since \( p(i) = -2 \neq 0 \) we have that
\[
wp = \frac{w}{p(a)} = e^{\frac{\theta}{2}} = -\frac{1}{2}(\cos \theta + i \sin \theta)
\]
Therefore \( y_{p1} = \text{imaginary part of } wp = -\frac{1}{2} \sin \theta \)
To find a particular solution for \(-e^{2\theta}\) note that \( p(2) = (2)^2 - 1 = 3 \) so
\[
y_{p2} = \frac{e^{2\theta}}{3}
\]
A general solution is then
\[
y = y_h + y_{p1} + y_{p2} = c_1 e^{\theta} + c_2 e^{-\theta} - \frac{1}{2} \sin \theta - \frac{e^{2\theta}}{3}
\]

Method 2:
For the equation \( y'' - y = \sin \theta \)
\[
y_{p1} = A \cos \theta + B \sin \theta
\]
\[
(-A \cos \theta - B \sin \theta) - (A \cos \theta + B \sin \theta) = -2A \cos \theta - 2B \sin \theta = \sin \theta
\]
\[-2A = 0
\]
\[-2B = 1
\]
\[A = 0, B = -1/2
\]
So, \( y_{p1} = -(1/2) \sin \theta \)
For the equation \( y'' - y = e^{2\theta} \)
\[
y_{p2} = Ae^{2\theta}
\]
\[
3 Ae^{2\theta} = e^{2\theta} \Rightarrow A = 1/3
\]
\[
y_{p2} = (1/3)e^{2\theta}
\]
\[
y = y_h + y_{p1} + y_{p2} = c_1 e^{\theta} + c_2 e^{-\theta} - \frac{1}{2} \sin \theta - \frac{e^{2\theta}}{3}
\]
The initial conditions imply since \( y' = c_1 e^{\theta} - c_2 e^{-\theta} - \frac{1}{2} \cos \theta - \frac{2}{3} e^{2\theta} \)
\[
y(0) = c_1 + c_2 - \frac{1}{3} = 1
\]
\[
y'(0) = c_1 - c_2 - \frac{1}{2} - \frac{2}{3} = -1
\]
\[
c_1 + c_2 = \frac{4}{3}
\]
\[
c_1 - c_2 = \frac{1}{6}
\]
Hence \( c_1 = \frac{3}{4}, c_2 = \frac{7}{12} \) Thus
\[
y = \frac{3}{4} e^{\theta} + \frac{7}{12} e^{-\theta} - \frac{1}{2} \sin \theta - \frac{e^{2\theta}}{3}
\]