MA 221 Homework Solutions
Due date: February 25, 2014

Section 4.5 pg. 187-188 1, 2b, 2b, 6, 8, 17, 21, 23, 27, 29
(Underlined Problems are to be handed in)

1.) Given that $y_1(t) = (1/4)\sin 2t$ is a solution to $y'' + 2y' + 4y = \cos 2t$ and that $y_2(t) = t/4 - 1/8$ is a solution to $y'' + 2y' + 4y = t,$ use the superposition principle to find soltuions of

(b) $y'' + 2y' + 4y = 2t - 3\cos 2t$

Since the right hand side of this last equation is twice the right hand side of the second equation given plus -3 times the right hand side of the first equation given a particular solution for (b) is

$2y_2 - 3y_1 = \frac{t}{2} - \frac{1}{4} - \frac{3}{4}\sin 2t$

2b) Given that $y_1(t) = \cos t$ is a solution to $y'' - y' + y = \sin t$ and that $y_2(t) = e^{2t}/3$ is a solution to $y'' - y' + y = e^{3t},$ use the superposition principle to find a solutions of

$$y'' - y' + y = \sin t - 3e^{2t}$$

$$y_p = \cos t - e^{2t}$$

In problems 6 & 8 a nonhomogeneous equation and a particular solution are given. Find a general solution for the equation.

6.) $\theta'' - \theta' - 2\theta = 1 - 2t \quad \theta_p(t) = t - 1$

The homogeneous equation is

$\theta'' - \theta' - 2\theta = 0$

Thus $r^2 - r - 2 = (r-2)(r+1) = 0$ so $r = 2, -1$

$y_h = c_1e^{2t} + c_2e^{-t}$

Therefore by the superposition principle:

$y = \theta_h + \theta_p = t - 1 + c_1e^{2t} + c_2e^{-t}$

8.) $y'' = 2y' - y + 2e^x \quad y_p(x) = x^2e^x$

The homogeneous equation is

$y'' - 2y' + y = 0$

Thus $r^2 - 2r + 1 = (r-1)^2 = 0$ so $r = 1$ is a repeated root and

$y_h = c_1e^x + c_2xe^x$

Therefore

$y = y_h + y_p = c_1e^x + c_2xe^x + x^2e^x$

17.) $y'' - y = -11t + 1$

The indicial equation is $r^2 - 1 = 0$ so $r = \pm 1$ and

$$y_h = c_1e^t + c_2e^{-t}$$

Let

$$y_p = At + B$$
$$y_p' = A$$
$$y_p'' = 0$$

The DE implies

$$-At - B = -11t + 1$$

so $A = 11, B = -1$ and

$$y_p = 11t - 1$$

Thus

$$y = y_h + y_p = c_1 e^t + c_2 e^{-t} + 11t - 1$$

In problem 21, find a general solution to the differential equation.

21.) $y''(\theta) + 2y'(\theta) + 2y(\theta) = e^{-\theta}\cos\theta$

Let $p(r) = r^2 + 2r + 2 = 0$

$r = -1 \pm i$

$y_h(\theta) = c_1 e^{-\theta}\cos\theta + c_2 e^{-\theta}\sin\theta = (c_1\cos\theta + c_2\sin\theta)e^{-\theta}$

We now find a particular solution.

Method 1:

$y_p(\theta) = \theta(A\cos\theta + B\sin\theta)e^{-\theta}$

$y_p'(\theta) = (A\cos\theta + B\sin\theta)e^{-\theta} + \theta[(A\cos\theta + B\sin\theta)e^{-\theta}]'$

$y_p''(\theta) = 2[(A\cos\theta + B\sin\theta)e^{-\theta}]' + \theta[(A\cos\theta + B\sin\theta)e^{-\theta}]'' = 2[(B-A)\cos\theta - (B+A)\sin\theta]$

$y_p'' + 2y_p' + 2y_p = 2[(B-A)\cos\theta - (B+A)\sin\theta]e^{-\theta} + 2(A\cos\theta + B\sin\theta)e^{-\theta} = 2(B\cos\theta - A\sin$

$A = 0, B = 1/2,$ so $y_p(\theta) = (1/2)\theta e^{-\theta}\sin\theta$

Method 2: Consider a companion $v''(\theta) + 2v'(\theta) + 2v(\theta) = e^{-\theta}\sin\theta.$ Multiply this equation by i and add it to the original equation. Letting $w = y + iv,$ we have

$$w'' + 2w' + 2w = e^{-\theta}(\cos\theta + i\sin\theta) = e^{-\theta}e^{i\theta} = e^{(i-1)\theta}$$

Then $p(i-1) = (i-1)^2 + 2(i-1) + 2 = -2i + 2i - 1 + 2 = 0.$ $p'(r) = 2r + 2$ so $p(i-1) = 2(i-1) + 2 = 2i$

Therefore

$$w_p = -\frac{\theta e^{(i-1)\theta}}{2} = \frac{\theta e^{-\theta}(\cos\theta + i\sin\theta)}{2i} = -\frac{1}{2}\theta e^{-\theta}(i\cos\theta - \sin\theta)$$

Thus

$$y_p = \operatorname{Re} w_p = \frac{1}{2}\theta e^{-\theta}\sin\theta$$

Therefore

$$y = y_h + y_p = c_1 e^{-\theta}\cos\theta + c_2 e^{-\theta}\sin\theta + \frac{1}{2}\theta e^{-\theta}\sin\theta$$

23.) $y' - y = 1, \; y(0) = 0$

This is a first order linear equation. Multiply by $e^{\int -dx} = e^{-x}$. The DE becomes

$$\frac{d}{dx}(e^{-x}y) = e^{-x}$$

so

$$e^{-x}y = -e^{-x} + c$$

or

$$y = -1 + ce^{x}$$

Then

$$y(0) = -1 + c = 0$$

and $c = 1$

$$y = -1 + e^{x}$$

In problems 27 and 29, find the solution to the Initial Value Problem.

27.) $y''(x) - y'(x) - 2y(x) = \cos x - \sin 2x; \qquad y(0) = -7/20, \qquad y'(0) = 1/5$

$p(r) = r^2 - r - 2 = 0 = (r-2)(r+1) \Rightarrow r = -1; r = 2$

$y_h(x) = c_1e^{-x} + c_2e^{2x}$

Method 1:

To find a particular solution for $\cos x$ we consider the two equations

$y'' - y' - 2y = \cos x$

$v'' - v' - 2v = \sin x$

Multiply the second equation by i and add it to the first eqaution and let $w = y + iv$ to get

$w'' - w' - 2w = \cos\theta + i\sin\theta = e^{i\theta}$

Since $p(i) = -1 - i - 2 = -3 - i \neq 0$ we have that

$$\begin{aligned} w_p &= \frac{ke^{\alpha\theta}}{p(\alpha)} = \frac{e^{i\theta}}{-3-i} = -\frac{1}{3+i}\frac{(3-i)}{(3-i)}(\cos\theta + i\sin\theta) \\ &= \frac{i-3}{10}(\cos\theta + i\sin\theta) \end{aligned}$$

Therefore y_{p_1} =real part of $w_p = -(3/10)\cos x - (1/10)\sin x$

To find a particular solution for $-\sin 2x$ we consider the two equations

$y'' - y' - 2y = -\sin 2x$

$v'' - v' - 2v = -\cos 2x$

Multiply the second equation by i and add it to the first eqaution and let $w = v + iy$ to get

$w'' - w' - 2w = -\cos 2x - i\sin 2x = -e^{2ix}$

Since $p(2i) = -4 - 2i - 2 = -6 - 2i \neq 0$ we have that

$$\begin{aligned} w_p &= \frac{ke^{\alpha\theta}}{p(\alpha)} = \frac{-e^{2ix}}{-6-2i} = \frac{1}{6+2i}\frac{(6-2i)}{(6-2i)}(\cos 2x + i\sin 2x) \\ &= \frac{2(3-i)}{40}(\cos 2x + i\sin 2x) \end{aligned}$$

Therefore y_{p_2} =imaginary part of $w_p = -(1/20)\cos 2x + (3/20)\sin 2x$

$$y = y_h + y_{p_1} + y_{p_2} = c_1e^{-x} + c_2e^{2x} - \frac{3}{10}\cos x - \frac{1}{10}\sin x - \frac{1}{20}\cos 2x + \frac{3}{20}\sin 2x$$

Method 2:
For the equation $y'' - y' - 2y = \cos x$
$y_{p,1}(x) = A\cos x + B\sin x$

$$y'' - y' - 2y = (-A\cos x - B\sin x) - (-A\sin x + B\cos x) - 2(A\cos x + B\sin x)$$
$$= (-3A - B)\cos x + (A - 3B)\sin x = \cos x$$

$A = -3/10; B = -1/10$
So, $y_{p_1}(x) = -(3/10)\cos x - (1/10)\sin x$
For the equation $y'' - y' - 2y = -\sin 2x$
$y_{p_2}(x) = A\cos 2x + B\sin 2x$

$$y'' - y' - 2y = (-4A\cos 2x - 4B\sin 2x) - (-2A\sin 2x + 2B\cos 2x) - 2(A\cos 2x + B\sin 2x)$$
$$= (-6A - 2B)\cos 2x + (2A - 6B)\sin 2x = -\sin 2x$$

$A = -1/20; B = +3/20$
So, $y_{p_2}(x) = -(1/20)\cos 2x + (3/20)\sin 2x$
so again we have

$$y = y_h + y_{p_1} + y_{p_2} = c_1e^{-x} + c_2e^{2x} - \frac{3}{10}\cos x - \frac{1}{10}\sin x - \frac{1}{20}\cos 2x + \frac{3}{20}\sin 2x$$
$$y' = -c_1e^{-x} + 2c_2e^{2x} + \frac{3}{10}\sin x - \frac{1}{10}\cos x + \frac{1}{10}\sin 2x + \frac{3}{10}\cos 2x$$

Next, we find c_1 and c_2 such that the initial conditions are satisfied.

$$y(0) = c_1 + c_2 - \frac{3}{10} - \frac{1}{20} = -\frac{7}{20}$$
$$y'(0) = -c_1 + 2c_2 - \frac{1}{10} + \frac{3}{10} = \frac{1}{5}$$

Thus

$$c_1 + c_2 = 0$$
$$-c_1 + 2c_2 = 0$$

Hence $c_1 = c_2 = 0$
With these constants the solution is,

$$y(x) = -\frac{3}{10}\cos x - \frac{1}{10}\sin x - \frac{1}{20}\cos 2x + \frac{3}{20}\sin 2x$$

29.) $y''(\theta) - y(\theta) = \sin\theta - e^{2\theta}$ $y(0) = 1,$ $y'(0) = -1$
The characteristic equation is $p(r) = r^2 - 1 = 0,$ so $r = \pm 1$ and $y_h = c_1e^{\theta} + c_2e^{-\theta}$.

Method 1:
To find a particular solution for $\sin\theta$ we consider the two equations
$v'' - v = \cos\theta$
$y'' - y = \sin\theta$
Multiply the second equation by i and add it to the first eqaution and let $w = v + iy$ to get
$w'' - w = \cos\theta + i\sin\theta = e^{i\theta}$

Since $p(i) = -2 \neq 0$ we have that
$w_p = \frac{ke^{\alpha\theta}}{p(\alpha)} = \frac{e^{i\theta}}{-2} = -\frac{1}{2}(\cos\theta + i\sin\theta)$
Therefore y_{p_1} =imaginary part of $w_p = -\frac{1}{2}\sin\theta$
To find a particular solution for $-e^{2\theta}$ note that $p(2) = (2)^2 - 1 = 3$ so
$y_{p_2} = \frac{-e^{2\theta}}{3}$
A general solution is then
$y = y_h + y_{p_1} + y_{p_2} = c_1e^{\theta} + c_2e^{-\theta} - \frac{1}{2}\sin\theta - \frac{e^{2\theta}}{3}$

Method 2:
For the equation $y'' - y = \sin\theta$
$y_{p,1} = A\cos\theta + B\sin\theta$
$(-A\cos\theta - B\sin\theta) - (A\cos\theta + B\sin\theta) = -2A\cos\theta - 2B\sin\theta = \sin\theta$
$-2A = 0$
$-2B = 1$
$A = 0, B = -1/2$
So, $y_{p,1} = -(1/2)\sin\theta$
For the equation $y'' - y = e^{2\theta}$
$y_{p,2} = Ae^{2\theta}$
$3Ae^{2\theta} = e^{2\theta} \Longrightarrow A = 1/3$
$y_{p,2} = (1/3)e^{2\theta}$
$y = y_h + y_{p_1} + y_{p_2} = c_1e^{\theta} + c_2e^{-\theta} - \frac{1}{2}\sin\theta - \frac{e^{2\theta}}{3}$

The initial conditions imply since $y' = c_1e^{\theta} - c_2e^{-\theta} - \frac{1}{2}\cos\theta - \frac{2}{3}e^{2\theta}$
$y(0) = c_1 + c_2 - \frac{1}{3} = 1$
$y'(0) = c_1 - c_2 - \frac{1}{2} - \frac{2}{3} = -1$
$c_1 + c_2 = \frac{4}{3}$
$c_1 - c_2 = \frac{1}{6}$
Hence $c_1 = \frac{3}{4}$, $c_2 = \frac{7}{12}$ Thus
$y = \frac{3}{4}e^{\theta} + \frac{7}{12}e^{-\theta} - \frac{1}{2}\sin\theta - \frac{e^{2\theta}}{3}$