MA 221 Homework Solutions
Due date: February 27, 2014

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(Underlined problems are to be handed in)

Section 4.6 pg 193

2.) Find a general solution to the differential equation using variation of parameters.

$y'' + 4y = \tan 2t$

This equation has associated homogenous equation

$y'' + 4y = 0$

The roots of the associated homogenous equation, $r^2 + 4 = 0$, are $r = \pm 2i$. Therefore, a general solution to this equation is

$y_h(t) = c_1 \cos 2t + c_2 \sin 2t$

For the variation of parameters method, we let

$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ where $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$

Thus, $y_1'(t) = -2\sin 2t$ and $y_2'(t) = 2\cos 2t$. This means that we have to solve the system

$v_1' \cos 2t + v_2' \sin 2t = 0$

$-2v_1'(t)\sin 2t + 2v_2'(t)\cos 2t = \tan 2t$

Multiplying the first equation by $s \sin 2t$ the second equation by $\frac{1}{2}\cos 2t$ and adding the resulting equations together, we get:

$v_2'(t) = \frac{1}{2}\sin 2t$

$v_2 = \frac{1}{2}\int \sin 2t dt = -\frac{1}{4}\cos 2t + c_3$

Substituting v_2' into the first equation($v_1' \cos 2t + v_2' \sin 2t = 0$), we get

$v_1'(t) = -v_2(t)\tan 2t = -\frac{1}{2}\frac{\sin^2 2t}{\cos 2t} = -\frac{1}{2}\frac{1-\cos^2 2t}{\cos t}$

$= -\frac{1}{2}(\cos 2t - \sec 2t)$

$\Rightarrow v_1(t) = \frac{1}{2}\int(\cos 2t - \sec 2t)dt = \frac{1}{4}(\sin 2t - \ln|\sec 2t + \tan 2t|) + c_4$

We take $c_3 = c_4 = 0$ since we just need a particular solution. Thus,

$y_p(t) = \frac{1}{4}(\sin 2t - \ln|\sec 2t + \tan 2t|)\cos 2t - \frac{1}{4}\cos 2t \sin 2t$

$= -\frac{1}{4}\cos 2t \ln|\sec 2t + \tan 2t|$

and general solution is

$y_p(t) = y_h(t) + y_p(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{4}\cos 2t \ln \sec 2t + \tan 2t$

11.) Find a general solution to the differential equation

$$y'' + y = \tan^2 t$$

Two independent solutions to the corresponding homogenous equation, $y'' + y = 0$, are $y_1(t) = \cos t$ and $y_2(t) = \sin t$. A particular solution to the original equation is of the form

$y_p(t) = v_1(t)\cos t + v_2(t)\sin t,$

where $v_1(t)$ and $v_2(t)$ satisfy

$v_1'(t)\cos t + v_2'(t)\sin t = 0$
$-v_1'(t)\sin t + v_2'(t)\cos t = \tan^2 t,$

Multiplying the first equation by $\sin t$ and the second equation by $\cos t$, and adding them together yield

$v_2'(t) = \tan^2 t\cos t = (\sec^2 t - 1)\cos t = \sec t - \cos t$

We find $v_1'(t)$ from the first equation in the system.

$v_1'(t) = -v_2'(t)\tan t = -(\sec t - \cos t)\tan t = \sin t - \frac{\sin t}{\cos^2 t}$

Integrating, we get

$v_1(t) = \int(\sin t - \frac{\sin t}{\cos^2 t})dt = -\cos t - \sec t$
$v_2(t) = \int(\sec t - \cos t)dt = \ln|\sec t + \tan t| - \sin t$

where we take zero integrating constants. Therefore,

$y_p(t) = -(\cos t + \sec t)\cos t + (\ln|\sec t + \tan t| - \sin t)\sin t = \sin t \ln|\sec t + \tan t| - 2,$

and a general solution is given by

$y(t) = c_1\cos t + c_2\sin t + \sin t\ln|\sec t + \tan t| - 2.$

18.) Find a general solution to the DE $y'' - 6y' + 9y = t^{-3}e^{3t}$

The characteristic equation is $p(r) = r^2 - 6r + 9 = (r-3)^2 = 0$ so $r = 3$ is a repeated root and

$y_h = c_1e^{3t} + c_2te^{-3t}$

Hence $y_p = v_1e^{3t} + v_2te^{3t}$, and we have

$v_1'e^{3t} + v_2'te^{3t} = 0$
$3v_1'e^{3t} + v_2'(3te^{3t} + e^{3t}) = \frac{e^{3t}}{t^3}$

The equations for v_1' and v_2' are

$$v_1' = \frac{\begin{vmatrix} 0 & te^{3t} \\ \frac{e^{3t}}{t^3} & 3te^{3t} + e^{3t} \end{vmatrix}}{\begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & 3te^{3t} + e^{3t} \end{vmatrix}} = \frac{-\frac{e^{6t}}{t^2}}{e^{6t}} = -\frac{1}{t^2}$$

$$v_2' = \frac{\begin{vmatrix} e^{3t} & 0 \\ 3e^{3t} & \frac{e^{3t}}{t^3} \end{vmatrix}}{\begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & 3te^{3t} + e^{3t} \end{vmatrix}} = \frac{\frac{e^{6t}}{t^3}}{e^{6t}} = \frac{1}{t^3}$$

$v_1 = t^{-1}$
$v_2 = -\frac{1}{2}t^{-2}$
$y_p = v_1e^{3t} + v_2te^{3t}$
$= t^{-1}e^{3t} - \frac{1}{2}t^{-1}e^{3t} = \frac{1}{2}t^{-1}e^{3t}$

Hence

$y = y_h + y_p = c_1e^{3t} + c_2te^{-3t} + \frac{1}{2}t^{-1}e^{3t}$

SNB check: $y'' - 6y' + 9y = t^{-3}e^{3t}$, Exact solution is: $y(t) = \frac{1}{2t}e^{3t} + C_1e^{3t} + C_2e^{3t}t$

Section 8.5 pg 454

1.) Use the substitution $y = x^r$ to find a general solution to the equation

$x^2y''(x) + 6xy'(x) + 6y(x) = 0$

$y(x) = x^r$

$y'(x) = rx^{r-1}$

$y''(x) = r(r-1)x^{r-2}$

$x^2r(r-1)x^{r-2} + 6xrx^{r-1} + 6y = 0$

$(r^2 + 5r + 6) = 0$

$(r+3)(r+2) = 0$

$r = -2, r = -3$

$y(x) = c_1x^{-2} + c_2x^{-3}$

3.)Use the substitution $y = x^r$ to find a general solution to the equation

$x^2y''(x) - xy'(x) + 17y(x) = 0$

$y(x) = x^r$

$y'(x) = rx^{r-1}$

$y''(x) = r(r-1)x^{r-2}$

$x^2[r(r-1)x^{r-2}] - x(rx^{r-1}) + 17x^r = 0$

$x^r(r^2 - 2r + 17)$

$(r^2 - 2r + 17) = 0$

$r = \frac{2-\sqrt{2^2-4(1)(17)}}{2} = 1 \pm 4i$

$y_1(x) = x\cos(4\ln x), y_2(x) = x\sin(4\ln x)$

$y(x) = c_1x\cos(4\ln x) + c_2x\sin(4\ln x)$

5.) Use the substitution $y = x^r$ to find a general solution to $\frac{d^2y}{dx^2} = \frac{5}{x}\frac{dy}{dx} - \frac{13}{x^2}y$

Notice that, since $x > 0$, we can multiply this differential equation by x^2 and rewrite it to obtain

$x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 13y = 0$

$y(x) = x^r$

$y'(x) = rx^{r-1}$

$y''(x) = r(r-1)x^{r-2}$

$r(r-1)x^r - 5rx^r + 13x^r = 0$

$(r^2 - 6r + 13)x^r = 0$

$(r^2 - 6r + 13) = 0$

$r = \frac{6+-\sqrt{36-52}}{2} = 3 + -2i$

$y_1(x) = x^3\cos(2\ln x), y_2(x) = x^3\sin(2\ln x)$

$y(x) = c_1x^3\cos(2\ln x) + c_2x^3\sin(2\ln x)$

15.) Solve the initial value problem

$t^2x(t) - 12x(t) = 0 \quad x(1) = 3, \; x'(1) = 5$

The indicial equation is

$r^2 + (p-1)r + q = r^2 - r - 12 = (r-4)(r+3) = 0$

Thus $r = 4, -3$ and

$x(t) = c_1t^4 + c_2t^{-3}$

The initial conditions imply since $x'(t) = 4c_1t^3 - 3c_2t^{-4}$

$x(1) = c_1 + c_2 = 3$

$x'(1) = 4c_1 - 3c_2 = 5$
Therefore $c_1 = 2, c_2 = 1$ so that
$x(t) = 2t^4 + t^{-3}$