MA 221 Homework Solutions
Due date: March 11/12, 2010

7.5 p. 409 #1, 3, 5, 6, 7, 15, 17, 19
(Underlined problems are to be handed in)
For problems 1, 3, 5, and 6, solve the given initial value problem using the method of
Laplace transforms.

1.)

\[ y'' - 2y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = 4 \]

Taking the Laplace transform of \( y'' - 2y' + 5y = 0 \) and applying the linearity of the Laplace transform yields:

\[ L\{y''\} - 2L\{y'\} + 5L\{y\} = L\{0\} \]

If we put \( Y(s) = L\{y\}(s) \) and apply Theorem 5 of Section 7.3 of the text, we get:

\[ L\{y'\} = sY(s) - y(0) = sY(s) - 2 \]

and

\[ L\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 4 \]

Substituting these expressions and using the fact that \( L\{0\} = 0 \) and solving for \( Y(s) \) yields

\[ [s^2Y(s) - 2s - 4] - 2[sY(s) - 2] + 5Y(s) = 0 \]

Solving for \( Y(s) \) gives

\[ Y(s) = \frac{2s}{s^2 - 2s + 5} = \frac{2s}{(s^2 - 2s + 1) + 4} \]

\[ = \frac{2s}{(s - 1)^2 + 2^2} \]

\[ = 2\left( \frac{s}{(s - 1)^2 + 2^2} \right) = 2\left( \frac{s - 1 + 1}{(s - 1)^2 + 2^2} \right) \]

\[ = 2\left( \frac{s - 1}{(s - 1)^2 + 2^2} + \frac{1}{(s - 1)^2 + 2^2} \right) \]

Using Table 7.1 on page 384 of the text to find the inverse Laplace transform of the above gives:

\[ y(t) = 2e^t \cos 2t + e^t \sin 2t \]

3)

\[ y'' + 6y' + 9y = 0 \quad y(0) = -1 \quad y'(0) = 6 \]

Proceeding as in the example above after taking Laplace transforms of both sides of the DE and using the initial conditions leads to

\[ [s^2Y(s) + s - 6] + 6[sY(s) + 1] + 9Y(s) = 0 \]

\[ (s^2 + 6s + 9)Y(s) + s = 0 \]
Thus
\[ Y(s) = -\frac{s}{s^2 + 6s + 9} = -\frac{s}{(s + 3)^2} \]

Now
\[ \frac{s}{(s + 3)^2} = \frac{A}{s + 3} + \frac{B}{(s + 3)^2} \]

Multiplying by \((s + 3)^2\) and setting \(s = -3\) yields \(B = -3\) so
\[ \frac{s}{(s + 3)^2} = \frac{A}{s + 3} + \frac{-3}{(s + 3)^2} \]

Letting \(s = 0\) in this equation gives
\[ 0 = \frac{A}{3} + \frac{-3}{9} \]

so \(A = 1\). Thus
\[ \frac{s}{(s + 3)^2} = \frac{1}{s + 3} - \frac{3}{(s + 3)^2} \]

and
\[ Y(s) = -\frac{s}{s^2 + 6s + 9} = -\frac{s}{(s + 3)^2} = \frac{-1}{s + 3} + \frac{3}{(s + 3)^2} \]

\[ y(t) = L^{-1}\left\{ \frac{-1}{s + 3} \right\} + L^{-1}\left( \frac{3}{(s + 3)^2} \right) \]
\[ = -e^{-3t} + 3t e^{-3t} \]

5.) \( w'' + w = t^2 + 2; \quad w(0) = 1, \quad w'(0) = -1 \)
\[ \mathcal{L}\{w''\}(s) + W(s) = \mathcal{L}\{t^2 + 2\}(s) = \mathcal{L}\{t^2\}(s) + 2 \mathcal{L}\{1\}(s) = \frac{2}{s^3} + \frac{2}{s} \]
Since \( \mathcal{L}\{w''\}(s) = s^2 W(s) - sw(0) - w'(0) = s^2 W(s) - s + 1 \), we have
\[ [s^2 W(s) - s + 1] + W(s) = \frac{2}{s^3} + \frac{2}{s} \quad \Rightarrow \quad (s^2 + 1) W(s) = s - 1 + \frac{2(s^2 + 1)}{s^3} \quad \Rightarrow \quad W(s) = \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1} \]

Now taking the inverse Laplace transform, we obtain
\[ w = \mathcal{L}^{-1}\left\{ \frac{s}{s^2 + 1} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 1} \right\} + \mathcal{L}^{-1}\left\{ \frac{2}{s^2} \right\} = \cos t - \sin t + t^2 \]

6.) \( y'' - 4y' + 5y = 4e^{3t}; \quad y(0) = 2, \quad y'(0) = 7 \)
Let \( Y = L\{y\} \). Then taking the Laplace transform of the equation and using linearity yields
\[ L\{y''\} - 4L\{y'\} + 5L\{y\} = L\{4e^{3t}\} \]
Solving for \( Y \) gives
\[ [s^2 Y - 2s - 7] - 4[sY - 2] + 5Y = \frac{4}{s^3} \]
\[ \rightarrow \quad (s^2 - 4s + 5)Y = 2s - 1 + \frac{4}{s^3} \]
\[ \rightarrow \quad Y = \frac{2s - 1 + \frac{4}{s^3}}{s^2 - 4s + 5} = \frac{2s^2 - 7s - 7}{(s^2 - 4s + 5)(s^2 - 4s + 5)} = \frac{2}{s - 3} + \frac{1}{(s - 2)^2 + 1} \]
Now, taking the inverse Laplace transform, we obtain
For problems 15 and 19, solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem.

15.) Solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem.

$$y'' - 3y' + 2y = \cos t; \quad y(0) = 0, \quad y'(0) = -1$$

Taking the Laplace transform and applying the linearity of the Laplace transform yields

$$L\{y''\} - 3L\{y'\} + 2L\{y\} = L\{\cos t\}$$

If we put $Y(s) = L\{y\}(s)$, we get

$$L\{y''\} = sY(s)$$

and

$$L\{y''\} = s^2Y(s) + 1$$

Combining, and knowing the fact that $L\{\cos t\} = \frac{s}{s^2+1}$, yields the equation

$$s^2Y(s) + 1 - 3sY(s) + 2Y(s) = \frac{s}{s^2+1}$$

Solving for $Y(s)$, we get

$$Y(s) = \frac{-s^2 + s - 1}{(s^2 + 1)(s^2 - 3s + 2)} = \frac{-s^2 + s - 1}{(s^2+1)(s-1)(s-2)}$$

19.) $y'' + 5y' - y = e^t - 1$

$y(0) = 1, y'(0) = 1$

$Y(s) := \mathcal{L}\{y\}(s)$

$\mathcal{L}\{y''\}(s) = s^2Y(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\{y'\}(s) = sY(s) - sy(0) - y'(0) = s^2Y(s) - s - 1$

The Laplace transform applied to both sides of the equation, yields

$$[s^2Y(s) - s - 1] + 5[sY(s) - 1] - Y(s) = \mathcal{L}\{e^t\}(s) - \mathcal{L}\{1\}(s) = \frac{1}{s-1} - \frac{1}{s} = \frac{1}{s(s-1)} \Rightarrow$$

$$(s^2 + 5s - 1)Y(s) = \frac{1}{s(s-1)} + s + 6 = \frac{s^3 + 5s^2 - 6s + 1}{s(s-1)}$$

$$Y(s) = \frac{s^3 + 5s^2 - 6s + 1}{s(s-1)(s^2 + 5s - 1)}$$