

MA 221 Homework Solutions

Due date: March 18, 2014

7.4 pg. 374 - 375 # 1, 3, 5, 7, 9, 12, 15, 17, 21, 23

(Underlined problems are to be handed in)

For problems 1,3, 5, 7, and 9, determine the inverse Laplace transform of the given function.

$$1.) \frac{6}{(s-1)^4}$$

From table 7.1

$$\frac{6}{(s-1)^4} = \frac{3!}{(s-1)^4}$$

is the Laplace transform of

$$e^{at}t^n$$

with

$$a = 1$$

$$n = 3$$

Therefore

$$L^{-1} = \left\{ \frac{6}{(s-1)^4} \right\}(t) = e^t t^3$$

$$3.) \frac{s+1}{s^2 + 2s + 10}$$

$$L^{-1} \left\{ \frac{s+1}{s^2 + 2s + 10} \right\}(t) = L^{-1} \left\{ \frac{s+1}{(s+1)^2 + 3^2} \right\}(t)$$

From table 7.1

$$L^{-1} \left\{ \frac{s+1}{(s+1)^2 + 3^2} \right\}$$

is the Laplace transform of

$$e^{at} \cos bt$$

with

$$a = -1$$

$$b = 3$$

Therefore

$$L^{-1} = \left\{ \frac{s+1}{(s+1)^2 + 3^2} \right\}(t) = e^{-t} \cos 3t$$

5.)

$$\begin{aligned} \frac{1}{s^2 + 4s + 8} \\ L^{-1}\left\{\frac{1}{s^2 + 4s + 8}\right\}(t) &= L^{-1}\left\{\frac{1}{(s+2)^2 + 2^2}\right\}(t) = \frac{1}{2}L^{-1}\left\{\frac{2}{(s+2)^2 + 2^2}\right\}(t) \\ &= \frac{1}{2}e^{-2t} \sin 2t \end{aligned}$$

7.)

$$\begin{aligned} \frac{2s + 16}{s^2 + 4s + 13} &= \frac{2s + 16}{(s+2)^2 + 3^2} = \frac{2(s+2)}{(s+2)^2 + 3^2} + \frac{4 \times 3}{(s+2)^2 + 3^2} \\ 2L^{-1}\left\{\frac{s+2}{(s+2)^2 + 3^2}\right\}(t) + 4L^{-1}\left\{\frac{3}{(s+2)^2 + 3^2}\right\}(t) \\ &= 2e^{-2t} \cos 3t + 4e^{-2t} \sin 3t \end{aligned}$$

9.)

$$\begin{aligned} \frac{3s - 15}{2s^2 - 4s + 10} &= \frac{3}{2} \cdot \frac{s - 5}{s^2 - 2s + 5} = \frac{3}{2} \cdot \frac{(s-1) - 4}{(s-1)^2 + 2^2} = \frac{\left(\frac{3}{2}\right)(s-1)}{(s-1)^2 + 2^2} - \frac{3 \times 2}{(s-1)^2 + 2^2} \\ \text{Therefore} \end{aligned}$$

$$\begin{aligned} \frac{3}{2}L^{-1}\left\{\frac{(s-1)}{(s-1)^2 + 2^2}\right\} - 3L^{-1}\left\{\frac{2}{(s-1)^2 + 2^2}\right\} \\ \frac{3}{2}e^t \cos 2t - 3e^t \sin 2t \end{aligned}$$

For problems 12, 15 and 17, determine the partial fraction expansions for the given rational function.

12.)

$$\begin{aligned} \frac{-s - 7}{(s+1)(s-2)} \\ \frac{-s - 7}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} = \frac{A(s-2) + B(s+1)}{(s+1)(s-2)} \end{aligned}$$

This implies

$$-s - 7 = A(s - 2) + B(s + 1)$$

Taking $s = -1$, and $s = 2$, we find A, B, and C, respectively.

$$s = -1 : 1 - 7 = A(-1 - 2) + B(-1 + 1) \Rightarrow -6 = -3A$$

$$A = 2$$

$$s = 2 : -2 - 7 = A(2 - 2) + B(2 + 1) \Rightarrow -9 = 3B$$

$$B = -3$$

$$\frac{-s - 7}{(s + 1)(s - 2)} = \frac{2}{s+1} - \frac{3}{s-2}$$

$$\frac{-s - 7}{(s + 1)(s - 2)} = \frac{2}{s+1} - \frac{3}{s-2}$$

15.)

$$\frac{8s - 2s^2 - 14}{(s + 1)(s^2 - 2s + 5)}$$

$$\frac{8s - 2s^2 - 14}{(s + 1)(s^2 - 2s + 5)} = \frac{A}{s+1} + \frac{B(s - 1) + C(2)}{(s - 1)^2 + 2^2} = \frac{A[(s - 1)^2 + 4] + [B(s - 1) + 2C](s + 1)}{(s + 1)[(s - 1)^2 + 4]}$$

This implies

$$-8s - 2s^2 - 14 = A[(s - 1)^2 + 4] + [B(s - 1) + 2C](s + 1)$$

Taking $s = -1$, $s = 1$, and $s = 0$ we find A, B, and C, respectively.

$$s = -1 : 8(-1) - 2(-1)^2 - 14 = A[(-1 - 1)^2 + 4]$$

$$A = -3$$

$$s = 1 : 8(1) - 2(1)^2 - 14 = A[(1 - 1)^2 + 4] + 2C(1 + 1)$$

$$C = 1$$

$$s = 0 : 8(0) - 2(0)^2 - 14 = A[(0 - 1)^2 + 4] + [B(0 - 1 + 2C)](0 + 1)$$

$$B = 1$$

$$\frac{-3}{s+1} + \frac{(s - 1) + 2}{(s - 1)^2 + 2^2}$$

17.)

$$\frac{3s + 5}{s(s^2 + s - 6)}$$

$$\frac{3s + 5}{s(s^2 + s - 6)} = \frac{3s + 5}{s(s - 2)(s + 3)} = \frac{A}{s} + \frac{B}{s - 2} + \frac{C}{s + 3}$$

$$3s + 5 = A(s - 2)(s + 3) + Bs(s + 3) + Cs(s - 2)$$

$$s = 0 : 5 = A(-2)3$$

$$A = -\frac{5}{6}$$

$$\begin{aligned}
s &= 2 : \quad 11 = B(2)5 \\
C &= \frac{11}{10} \\
S &= -3 : \quad -4 = C(-3)(-5) \\
B &= -\frac{4}{15} \\
\frac{3s+5}{s(s^2+s-6)} &= -\frac{5}{6s} + \frac{11}{10(s-2)} - \frac{4}{15(s+3)}
\end{aligned}$$

In problems 21 and 23 find $L^{-1}\{F\}$

21.)

$$F(s) = \frac{6s^2 - 13s + 2}{s(s-1)(s-6)}$$

Since the denominator contains only nonrepeated linear factors, the partial fractions decomposition has the following form

$$\frac{6s^2 - 13s + 2}{s(s-1)(s-6)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-6} = \frac{A(s-1)(s-6) + Bs(s-6) + Cs(s-1)}{s(s-1)(s-6)}$$

Evaluating both sides of the equation for $s = 0, s = 6, s = 1$:

$$s = 0 : 2 = A(-1)(-6) \Rightarrow 2 = 6A$$

$$A = \frac{1}{3}$$

$$s = 6 : 6s^2 - 13s + 2 = 30C = 140$$

$$C = \frac{14}{3}$$

$$s = 1 : \quad 6 - 13 + 2 = -5B = -5$$

$$B = 1$$

$$\frac{6s^2 - 13s + 2}{s(s-1)(s-6)} = \frac{1}{3} \frac{1}{s} + \frac{1}{s-1} + \frac{14}{3} \frac{1}{s-6}$$

$$\begin{aligned}
L^{-1} \left\{ \frac{6s^2 - 13s + 2}{s(s-1)(s-6)} \right\} &= L^{-1} \left\{ \frac{1}{3} \frac{1}{s} + \frac{1}{s-1} + \frac{14}{3} \frac{1}{s-6} \right\} \\
&= \frac{1}{3} L^{-1} \left\{ \frac{1}{s} \right\} + L^{-1} \left\{ \frac{1}{s-1} \right\} + \frac{14}{3} L^{-1} \left\{ \frac{1}{s-6} \right\} \\
&= \frac{1}{3} + e^t + \frac{14}{3} e^{6t}
\end{aligned}$$

23.)

$$F(s) = \frac{5s^2 + 34s + 53}{(s+3)^2(s+1)}$$

In this problem, the denominator of $F(s)$ has a simple linear factor, $s+1$, and a double linear factor, $s+3$. Thus the decomposition has the following form

$$\frac{5s^2 + 34s + 53}{(s+3)^2(s+1)} = \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{C}{s+1} = \frac{A(s+1) + B(s+3)(s+1) + C(s+3)^2}{(s+3)^2(s+1)}$$

Evaluating both sides of the equation for $s = -1, s = -3$:

$$s = -1 : 5 - 34 + 53 = 4C = 24$$

$$C = 6$$

$$s = -3 : 45 - 102 + 53 = -2A = -4$$

$$A = 2$$

$$s = 0 \text{ (To find B)} : 53 = A + 3B + 9C$$

$$53 = 2 + 3B + 9(6) \Rightarrow 53 - 54 - 2 = 3B$$

$$3B = 53 - 54 - 2 \Rightarrow B = -1$$

$$\frac{5s^2 + 34s + 53}{(s+3)^2(s+1)} = 2\frac{1}{(s+3)^2} - \frac{1}{s+3} + 6\frac{1}{s+1}$$

$$\begin{aligned} L^{-1}\left\{\frac{5s^2 + 34s + 53}{(s+3)^2(s+1)}\right\} &= L^{-1}\left\{2\frac{1}{(s+3)^2} - \frac{1}{s+3} + 6\frac{1}{s+1}\right\} \\ &= 2L^{-1}\left\{\frac{1}{(s+3)^2}\right\} - L^{-1}\left\{\frac{1}{s+3}\right\} + 6L^{-1}\left\{\frac{1}{s+1}\right\} \\ &= 2te^{-3t} - e^{-3t} + 6e^{-t} \end{aligned}$$