MA 221 Homework Solutions
Due date: March 27, 2014

8.3 p. 445 #1, 3, 5, 7, 11, 12, 15,
(Underlined problems are to be handed in)

In problems 1, 3, 5 and 7 Determine all the singular points of the given differential equations.

1.) $(x+1)y'' - x^2y' + 3y = 0$
Dividing the entire equation by $(x+1)$ yields

$$y'' - \frac{x^2}{x+1}y' + \frac{3}{x+1}y = 0$$

We then see:

$$P(y) = -\frac{x^2}{x+1} \qquad Q(y) = \frac{3}{x+1}$$

These are rational functions and so they are analytical everywhere except, perhaps, at zeros of their denominators. Solving $x+1 = 0$ we find that $x = -1$ which is at a point of infinite discontinuity for both functions. Consequently, $x = -1$ is the only singular point of the given equation.

3.) $(\theta^2 - 2)y'' + 2y' + \sin\theta y = 0$
Writing the equation in standard form yields

$$y'' + \frac{2}{\theta^2 - 2}y' + \frac{\sin\theta}{\theta^2 - 2}y = 0$$

and

$$P(\theta) = \frac{2}{\theta^2 - 2} \qquad Q(\theta) = \frac{\sin\theta}{\theta^2 - 2}$$

The singularities are therefore at $\theta = \pm\sqrt{2}$.

Find at least the first four nonzero terms in a power series expansion about $x = 0$ for a general solution to the given differential equation.

5.) $(t^2 - t - 2)x'' + (t+1)x' - (t-2)x = 0$

$$x'' + \frac{t+1}{t^2-t-2}x' - \frac{t-2}{t^2-t-2}x = 0$$

$$p(t) = \frac{t+1}{t^2-t-2} = \frac{t+1}{(t+1)(t-2)}$$

$$q(t) = \frac{t-2}{t^2-t-2} = \frac{t-2}{(t+1)(t-2)}$$

The point $t = -1$ is a removable singularity for $p(t)$ since, for $t \neq -1$, we can cancel $(t+1)$ term in the numerator and denominator, and so $p(t)$ becomes analytic at $t = -1$ if we set

$p(-1) := \lim_{t\to -1} p(t) = \lim_{t\to -1} \frac{1}{t-2} = -\frac{1}{3}$

At the point $t = 2, p(t)$ has infinite discontinuity. Thus $p(t)$ is analytic everywhere except $t = 2.$ Similarly, $q(t)$ is analytic everywhere except $t = -1.$ Therefore, the given equation has two singular points, $t = -1$ and $t = 2.$

7.) $(\sin x)y'' + (\cos x)y = 0$

Putting the equation in standard form we get:

$y'' + \frac{(\cos x)}{(\sin x)}y = 0$ Hence:

$p(x) = 0 \quad q(x) = \frac{(\cos x)}{(\sin x)} = \cot x$

Since the cotangent function is $\pm\infty$ at integer multiples of π, we see that $q(x)$ is not defined and , therefore not analytical at $n\pi$. Hence the differential equation is singular only at the points πn, where n is an integer.

In problems 11, 12, 15 and 17 find at least the first four non zero terms in a power series expansion about $x = 0$ for a general solution to the given differential equation.

11.) $y' + (x+2)y = 0$

The coefficient, $x+2$, is a polynomial, and so it is analytical everywhere. Therefore, $x = 0$ is an ordinary point on the given equation.

We seek the power series solutiin in the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

We now substitute the power series for y and y' into the given differential equation and obtain:

$$y' + (x+2)y = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + (x+2)\sum_{n=0}^{\infty} a_n x^n = 0$$

We want to be able to write the left-hand side of this equation as a single power seriess. This will allow us to find expressions for the coefficient of each power of $x.$ Therefore, we first need to shift the indices in each power series above so that they sum over the same powers of x. Thus, we let $k = n-1$ in the first summation and note that this means that $n = k+1$ and that $k = 0$ when $n = 1$. This yield

$$\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=0}^{\infty} 2a_k x^k + \sum_{k=1}^{\infty} a_{k-1}x^k = 0$$

$$\Rightarrow \left[a_1 + \sum_{k=1}^{\infty}(k+1)a_{k+1}x^k \right] + \left[2a_0 + \sum_{k=1}^{\infty} 2a_k x^k \right] + \sum_{k=1}^{\infty} a_{k-1}x^k = 0$$

$$\Rightarrow (a_1 + 2a_0) + \sum_{k=1}^{\infty}[(k+1)a_{k+1} + 2a_k + a_{k-1}]x^k = 0$$

For the power series on the left hand side to be identically equato to zero, we must have all zero coefficients. Hence,

$(a_1 + 2a_0) = 0$

$(k+1)a_{k+1} + 2a_k + a_{k-1} = 0 \qquad$ for all $k \geq 1$

This yields:

$a_1 + 2a_0 = 0 \qquad \Rightarrow \qquad a_1 = -2a_0$

$k = 1 : 2a_2 + 2a_1 + a_0 = 0 \Rightarrow \qquad a_2 = \dfrac{(-2a_1 - a_0)}{2} = \dfrac{(4a_0 - a_0)}{2} = \dfrac{3a_0}{2}$

$k = 2 : 3a_3 + 2a_2 + a_0 = 0 \Rightarrow \qquad a_3 = \dfrac{(-2a_2 - a_1)}{3} = \dfrac{(-3a_0 + 2a_0)}{3} = \dfrac{-a_0}{3}$

Therefore,

$$y(x) = a_0 - 2a_0x + \frac{3a_0}{2}x^2 - \frac{a_0}{3}x^3 + \ldots\ldots = a_0\left(1 - 2x + \frac{3x^2}{2} - \frac{x^3}{3}\right)$$

12.) $\quad y' - y = 0$

The coefficient of y is the integer -1, which is analytic everywhere. Thus we expect to find a power series solution of the form

$y(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n$

Our task is to determine the coefficients a_n.

For this purpose we need the expansion for $y'(x)$ that is given by termwise differentiation of the above equation:

$y'(x) = 0 + a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}$.

We now substitute the series expansion for y and y' and obtain:

$\sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=0}^{\infty} a_nx^n = 0.$

We want to be able to write the left-hand side of this equation as a single power series. This will allow us to find expressions for the coefficient of each power of x. Therefore, we first need to shift the indices in each power series above so that they sum over the same powers of x. Thus, we let $k = n - 1$ in the first summation and note that this means that $n = k + 1$ and that $k = 0$ when $n = 1$. This yields

$\sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k$

In the second power series we need only to replace n with k. Substituting all of these expressions into their appropriate places yields

$\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k - \sum_{k=0}^{\infty} a_kx^k = 0.$

In order for this power series to equal to zero, each coefficient must be zero. Therefore, we obtain

$(k+1)a_{k+1} - a_k = 0 \qquad \rightarrow \qquad a_{k+1} = \frac{a_k}{(k+1)}$

Setting $k = 1, 2, 3\ldots$ and using the fact that $a_1 = a_0$

$a_2 = \frac{a_1}{(1+1)} = \frac{a_0}{2} \qquad\qquad a_4 = \frac{a_3}{(3+1)} = \frac{a_3}{4} = \frac{1}{4}(\frac{1}{3}(\frac{a_0}{2}))$

$a_3 = \frac{a_2}{(2+1)} = \frac{1}{3}(\frac{a_0}{2}) \qquad a_5 = \frac{a_4}{(4+1)} = \frac{a_4}{5} = \frac{1}{5}(\frac{1}{4}(\frac{1}{3}(\frac{a_0}{2}))) \qquad etc.$

Hence the power series for the solution takes the form

$y(x) = a_0(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots) \qquad = a_0\sum_{n=0}^{\infty}\frac{x^n}{n!} = a_0e^x$

15.)

$$y'' + (x-1)y' + y = 0$$

Here $P = x - 1$ and $Q = 1$ so there are no singularities and $x = 0$ is an ordinary point. Then

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

and the DE implies

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n (n) x^n - \sum_{n=1}^{\infty} a_n (n) x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

We shift the first and third sums above by letting $k = n-2$ or $n = k+2$ and $j = n-1$ or $n = j+1$ and get

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - \sum_{j=0}^{\infty} a_{j+1}(j+1)x^j + \sum_{n=1}^{\infty} a_n (n+1) x^n + a_0 = 0$$

Replacing all of the place keepers by m and writing out the first terms of the first and second sums leads to

$$2(1)a_2 - a_1 + a_0 + \sum_{m=1}^{\infty} [a_{m+2}(m+2)(m+1) - a_{m+1}(m+1) + a_m(m+1)]x^m = 0$$

Thus

$$2a_2 - a_1 + a_0 = 0$$
$$a_{m+2}(m+2)(m+1) - a_{m+1}(m+1) + a_m(m+1) = 0$$

or

$$a_{m+2} = \frac{a_{m+1} - a_m}{m+2} \quad m = 1, 2, 3, \ldots$$

Hence

$$a_2 = \frac{a_1 - a_0}{2}$$
$$m = 1 \Rightarrow a_3 = \frac{a_2 - a_1}{3} = \frac{\frac{a_1 - a_0}{2} - a_1}{3} = \frac{-(a_1 + a_0)}{6}$$
$$m = 2 \Rightarrow a_4 = \frac{a_3 - a_2}{4} = \frac{\frac{-(a_1 + a_0)}{6} - \frac{a_1 - a_0}{2}}{4} = \frac{-2a_1 + a_0}{12}$$

Therefore

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= a_0\left(1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \cdots\right) + a_1\left(x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{6} + \cdots\right)$$