MA 221 Homework Solutions
Due date: April 10, 2014

Section 10.4 page 598 - 599 # 5, 7, 12, 13

In problems 5 and 7 compute the Fourier sine series for the given function.
5) $f(x) = -1$ on $(0,1)$ so $L = 1$

If $f(x)$ is a function defined on $[0,L]$, then its Fourier sine expansion is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where} \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Here we have

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \text{ and } a_n = \frac{2}{1}\int_0^1 (-1)\sin(n\pi x)dx$$

$$a_n = \frac{2}{1}\int_0^1 (-1)\sin(n\pi x)dx = \frac{2}{n\pi}\cos(n\pi x)\Big|_0^1$$

$$= \frac{2}{n\pi}(\cos n\pi - \cos 0) = \frac{2[(-1)^n - 1]}{n\pi}$$

$$= \begin{cases} a_{2k} = 0 \text{ if } n = 2k \text{ is even} \\ a_{2k-1} = \frac{2\left[(-1)^{2k-1}-1\right]}{(2k-1)\pi} \text{ if } n = 2k-1 \text{ is odd} \end{cases}$$

$$= \begin{cases} a_{2k} = 0 \text{ if } n = 2k \text{ is even} \\ a_{2k-1} = -\frac{4}{(2k-1)\pi} \end{cases}$$

Hence

$$f(x) = -\frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)\sin(2k-1)\pi x$$

7) $f(x) = x^2 \quad 0 < x < \pi$

Here $L = \pi$ so

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad \text{where} \quad a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin(nx)dx$$

To compute this integral we integrate by parts twice

$$\left(\frac{\pi}{2}\right) a_n = \int_0^{\pi} x^2 \sin(nx)dx$$

$$= -x^2 \frac{\cos nx}{n}\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\cos nx dx$$

$$= -\frac{\pi^2 \cos n\pi}{n} + 0 + \frac{2}{n}\left[x\frac{\sin nx}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx dx\right]$$

$$= -\frac{\pi^2 \cos n\pi}{n} + \frac{2}{n}\left[0 - \frac{1}{n}\left(-\frac{\cos nx}{n}\right)\Big|_0^{\pi}\right]$$

$$= -\frac{\pi^2 \cos n\pi}{n} + \frac{2}{n^3}(\cos n\pi - 1) \quad n = 1.2.3.\ldots$$

Since $\cos n\pi = 1$ if n is even and -1 if n is odd

$$\left(\frac{\pi}{2}\right) a_n = -\frac{\pi^2(-1)^n}{n} + \frac{2}{n^3}[(-1)^n - 1] \quad n = 1.2.3.\ldots$$

Thus

$$a_n = \frac{2\pi(-1)^{n+1}}{n} + \frac{4[(-1)^n - 1]}{\pi n^3}$$

Therefore the Fourier sine series for $f(x) = x^2$ on $(0,\pi)$ is given by

$$\sum_{n=1}^{\infty}\left\{\frac{2\pi(-1)^{n+1}}{n} + \frac{4[(-1)^n - 1]}{\pi n^3}\right\}\sin nx$$

12) Find the Fourier cosine series for $f(x) = 1 + x$ on $0 < x < \pi$.

Here we use the formulas given in class. Namely,

$$f(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) = b_0 + \sum_{1}^{\infty} b_n \cos nx,$$

since the function is given on $[0,L] \Rightarrow L = \pi$.

$$b_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{\pi}\int_0^{\pi}(1+x)dx = \frac{1}{\pi}\left[\frac{x^2}{2} + x\right]_0^{\pi} = \frac{1}{2}\pi + 1$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{\pi}\int_0^{\pi}(1+x)\cos(nx)dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}\sin nx + \frac{1}{n^2}(\cos nx + nx\sin nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}[\cos n\pi - 1] = \frac{2}{\pi n^2}[(-1)^n - 1]$$

Note that if $n = 2k$ is even then $b_{2k} = 0$, If $n = 2k+1$ is odd, then

$$b_{2k+1} = \frac{2}{\pi(2k+1)^2}\left[(-1)^{2k+1} - 1\right] = -\frac{4}{\pi(2k+1)^2}$$

Thus

$$f(x) = \frac{1}{2}\pi + 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)x$$

13) Find the Fourier cosine series for $f(x) = e^x$ on $0 < x < 1$.

Here we use the formulas given in the book. Namely

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) \quad a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{n\pi x}{T}\right)dx$$

From a table of integrals we have

$$\int e^x \cos(n\pi x)dx = \frac{1}{\pi^2 n^2 + 1}(e^x \cos\pi nx + \pi n e^x \sin\pi nx)$$

For $n \geq 1$ we have

$$a_n = \frac{2}{\pi^2 n^2 + 1}(e^x \cos\pi nx + \pi n e^x \sin\pi nx)\Big|_0^1$$

$$= \frac{2e\cos n\pi}{\pi^2 n^2 + 1} - \frac{2(1)}{\pi^2 n^2 + 1} = \frac{2[(-1)^n e - 1]}{\pi^2 n^2 + 1}$$

Thus the Fourier cosine series for e^x on $0 < x < 1$ is

$$e - 1 + 2\sum_{n=1}^{\infty}\frac{[(-1)^n e - 1]}{\pi^2 n^2 + 1}\cos n\pi x$$

Sec 10.2 Problems #19, 20, 22
Sec 10.6 Problem #3

Section 10.2

Solve the vibrating string problem with $\alpha = 3$, $L = \pi$, and the given initial functions $f(x)$ and $g(x)$.

19.) $f(x) = 3\sin 2x + 12\sin 13x, \qquad g(x) = 0$

By letting $\alpha = 3$ and $L = \pi$ in formula (24) on page 585 of the text, we see that the solution we want will have the form

$u(x,t) = \sum_{n=1}^{\infty}[a_n \cos 3nt + b_n \sin 3nt]\sin nx$

Therefore, we see that

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}[-3na_n \sin 3nt + 3nb_n \cos 3nt]\sin nx$

In order for the solution to satisfy the initial conditions, we must find a_n and b_n such that

$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = 3\sin 2x + 12\sin 13x, \qquad (1)$

and

$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} 3nb_n \sin nx = 0.$

From the first condition, we observe that we must have a term for $n = 2, 13$ and for these terms we want $a_2 = 3$ and $a_{13} = 12$.

All of the other a_n's must be zero. By comparing coefficients in the second condition, we see that all b_n's must also be zero.

Therefore, by substituting these values into equation (1) above, we obtain the solution of the vibrating string problem with

$\alpha = 3$, $L = \pi$ and $f(x)$ and $g(x)$ as given. This solution is given by

$u(x,t) = 3[\cos(3)(2)t]\sin 2x + 12[\cos(3)(13)t]\sin 13x + 0$

Or by simplifying, we obtain

$$u(x,t) = 3\cos 6t \sin 2x + 12\cos 39t \sin 13x$$

20.) $f(x) = 0, \qquad g(x) = -2\sin 3x + 9\sin 7x - \sin 10x$

By letting $\alpha = 3$ and $L = \pi$ in formula (24) on page 585 of the text, we see that the solution we want will have the form

$u(x,t) = \sum_{n=1}^{\infty}[a_n \cos 3nt + b_n \sin 3nt]\sin nx$

Therefore, we see that

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}[-3na_n \sin 3nt + 3nb_n \cos 3nt]\sin nx$

In order for the solution to satisfy the initial conditions, we must find a_n and b_n such that

$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = 0$

and

$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} 3nb_n \sin nx = -2\sin 3x + 9\sin 7x - \sin 10x \qquad (2)$

From the first condition, we observe that $a_n = 0$

By comparing coefficients in the second condition, we see that we require

$9b_3 = -2$ or $b_3 = -\frac{2}{9}, \qquad 21b_7 = 9$ or $b_7 = \frac{3}{7}$

$30b_{10} = -2$ or $b_{10} = -\frac{1}{30}$ and all other b_n values must be zero. Therefore, by substituting these values into equation (2) above, we obtain the

solution of the vibrating string problem with $\alpha = 3$, $L = \pi$ and $f(x)$ and $g(x)$ as given. This solution is given by

$$\boxed{u(x,t) = -\frac{2}{9}\sin 9t \sin 3x + \frac{3}{7}\sin 21t \sin 7x - \frac{1}{30}30t \sin 10x}$$

22.)

$f(x) = \sin x - \sin 2x + \sin 3x, \qquad g(x) = 6\sin 3x - 7\sin 5x$

By letting $\alpha = 3$ and $L = \pi$ in formula (24) on page 585 of the text, we see that the solution we want will have the form

$u(x,t) = \sum_{n=1}^{\infty}[a_n \cos 3nt + b_n \sin 3nt]\sin nx$

Therefore, we see that

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}[-3na_n \sin 3nt + 3nb_n \cos 3nt]\sin nx$

In order for the solution to satisfy the initial conditions, we must find a_n and b_n such that

$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = \sin x - \sin 2x + \sin 3x$

and

$\dfrac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} 3nb_n \sin nx = 6\sin 3x - 7\sin 5x \qquad (2)$

From the first condition, we observe that we must have a term for $n = 1,2,3$ and for these terms we want $a_1 = 1, a_2 = -1$ and $a_3 = 1$.

All of the other a_n's must be zero. By comparing coefficients in the second condition, we see that we require

$(3)(3)b_3 = 6$ or $b_3 = \frac{2}{3}$, $\qquad (3)(5)b_5 = -7$ or $b_5 = -\frac{7}{15}$

We also see that all other values for b_n must be zero. Therefore, by substituting these values into equation (2) above, we obtain the

solution of the vibrating string problem with $\alpha = 3$, $L = \pi$ and $f(x)$ and $g(x)$ as given. This solution is given by

$$\boxed{u(x,t) = \cos 3t \sin x - \cos 6t \sin 2x + \cos 9t \sin 3x + \frac{2}{3}\sin 9t \sin 3x - \frac{7}{15}\sin 15t \sin 5x}$$