MA 221 Homework Solutions
Due date: April 16, 2013

Section 10.2, page 577 #27 & 29
(Underlined Problems are to be handed in)

27.)
\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \]

with \( u(r, \theta) = R(r)T(\theta) \). Substituting into the PDE we get
\[ R''T + \frac{1}{r} R'T = -\frac{1}{r^2} RT'' \]
or
\[ \frac{r^2 R'' + r R'}{R} = -\frac{T''}{T} = \lambda \]
since the left hand side is purely a function of \( r \), and the right hand side is purely a function of \( \theta \).

Thus we have the two ODEs
\[ r^2 R'' + r R' - \lambda R = 0 \]
\[ T'' + \lambda T = 0 \]

29.)
\[ \frac{\partial u}{\partial t} = \beta(u_{xx} + u_{yy}) \]

with \( u(x, y, t) = X(x)Y(y)T(t) \). Substituting into the PDE we get
\[ XYT' = \beta(X''YT + XY''T) \]
or
\[ \frac{1}{\beta} \frac{T'}{T} = \frac{X''Y + XY''}{XY} \]

Since the left hand side depends only on \( t \), whereas the right hand side depends only on \( x \) and \( y \), then
\[ \frac{1}{\beta} \frac{T'}{T} = \frac{X''Y + XY''}{XY} = K \]

where \( K \) is a constant. Therefore the equation for \( T(t) \) is
\[ T' - \beta KT = 0 \]

Now we also have from the above equation that
\[ X''Y + XY'' = KXY \]
or
$$\frac{X''}{X} = -\frac{Y''}{Y} + K$$

Since the left hand side depends only on x, whereas the right hand side depends only on y, we have

$$\frac{X''}{X} = -\frac{Y''}{Y} + K = J$$

where J is a constant. Therefore, we get the equations

$$X'' - KX = 0$$
$$Y'' + (J - K)Y = 0$$

Section 10.6 Problems 5, 9, 11
(1nderlined problems are to be handed in)

5.) \(f(x) = h_0 x/a,\ 0 < x \leq a,\)
\(f(x) = h_0 (L - x)/(L - a),\ a < x < L\)
and \(g(0) = 0\)

The problem is consistent because
\(g(0) = 0 = g(L)\) and \(f(0) = 0 = f(L)\)

The formal solution is given in equation (5) on page 625 of the text with the coefficients given in equations (6) and (7) on page 626. By equation (7)

\(g(x) = 0 = \sum_{n=1}^{\infty} b_n \left( \frac{na}{L} \right) \sin \left( \frac{na}{L} \right).\)

Thus, each term in this infinite series must be zero and so \(b_n = 0\) for all \(n\)’s. Therefore, the formal solution given in equation (5) on page 625 becomes

\(u(x,t) = \sum_{n=1}^{\infty} a_n \cos \left( \frac{na}{L} \right) t \sin \left( \frac{na}{L} \right)\)

Using equation (7) on page 609 for \(n = 1, 2, 3, \ldots\) we have

\(a_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{nx}{L} \right) dx = \frac{2}{L} \left[ h_0 \int_0^a x \sin \left( \frac{na}{L} \right) dx + h_0 \int_a^L \frac{L - x}{L - a} \sin \left( \frac{na}{L} \right) dx \right] = \frac{2h_0}{L} \left[ \frac{1}{a} \int_0^a x \sin \left( \frac{na}{L} \right) dx + \frac{L - a}{L - L/a} \int_a^L \sin \left( \frac{na}{L} \right) dx - \frac{1}{L - L/a} \int_a^L x \sin \left( \frac{na}{L} \right) dx \right]

By using integration by parts
\(\int x \sin \left( \frac{nx}{L} \right) dx = -\frac{x}{n\pi} \cos \left( \frac{nx}{L} \right) + \frac{1}{n^2\pi^2} \sin \left( \frac{nx}{L} \right)\)

For \(n = 1, 2, 3, \ldots\)

\(a_n = \frac{2h_0}{L} \left[ \frac{1}{a} \left[ -\frac{aL}{L^2} \cos \left( \frac{na}{L} \right) + \frac{L^2}{n^2\pi^2} \sin \left( \frac{na}{L} \right) \right] \right.\)
\(\left. - \frac{L^2}{n\pi(L-a)} \cos n\pi - \cos \left( \frac{nx}{L} \right) \right] - \frac{1}{L-a} \left[ -\frac{L^2}{n^2\pi^2} \cos n\pi + \frac{aL}{L^2} \cos \left( \frac{nx}{L} \right) \right] + \frac{L^2}{n^2\pi^2} \sin n\pi\)

\(u(x,t) = \frac{2h_0L^2}{\pi^2(a(L-a))} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{na}{L} \right) \cos \left( \frac{na}{L} \right) t \sin \left( \frac{na}{L} \right)\)

9.) \(\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0\)
\(u(0,t) = 0\) and \(\frac{\partial u}{\partial x} (L,t) = 0,\ t > 0\)
\(u(x,0) = f(x),\ 0 < x < L\)
\(u_t (x,0) = g(x),\ 0 < x < L\)
\(X(x)T''(t) = \alpha^2 X''(x)T(t)\)
\(T'(t) = \frac{x'(x)}{X(x)}\)
\(\frac{a^2 T''(t)}{T'(t)} = K \Rightarrow T'(t) - a^2 KT(t) = 0\)
\[
\frac{X''(x)}{X(x)} = K \Rightarrow X''(x) - KX(x) = 0
\]
We have the BCs \(X(0) = 0\) and \(X'(L) = 0\).
For a nontrivial solution for \(X(x)\) we need sines or cosines. Thus we set \(K = -\beta^2\), where \(\beta \neq 0\). Then
\[
X(x) = a \sin \beta x + b \cos \beta x
\]
Now \(X(0) = 0\) implies \(b = 0\), whereas \(X'(x) = a \beta \cos \beta x\) so \(X'(L) = 0\) implies that \(\cos \beta L = 0\). Thus
\[
\beta L = \left( \frac{2n+1}{2} \right) \pi, \quad n = 0, 1, 2, \ldots
\]
or
\[
\beta = \left( \frac{2n+1}{2} \right) \frac{\pi}{L}
\]
and
\[
X_n(x) = a_n \sin \left( \frac{2n+1}{2} \frac{\pi x}{L} \right), \quad n = 0, 1, 2, \ldots
\]
The equation for \(T(t)\) is
\[
T''(t) - \alpha^2 K T(t) = T'' + \alpha^2 \left( \left( \frac{2n+1}{2} \frac{\pi}{L} \right) \right)^2 T = 0
\]
so
\[
T_n(t) = b_n \sin \left( \left( \frac{2n+1}{2} \frac{\pi \alpha t}{L} \right) \right) + c_n \cos \left( \left( \frac{2n+1}{2} \frac{\pi \alpha t}{L} \right) \right), \quad n = 0, 1, 2, \ldots
\]
Therefore
\[
u(x, t) = \sum_{n=0}^{\infty} \left[ B_n \sin \left( \left( \frac{2n+1}{2} \frac{\pi a t}{L} \right) \right) + C_n \cos \left( \left( \frac{2n+1}{2} \frac{\pi a t}{L} \right) \right) \right] \sin \left( \frac{2n+1}{2} \frac{\pi x}{L} \right)
\]
The initial conditions imply
\[
u(x, 0) = f(x) = \sum_{n=0}^{\infty} C_n \sin \left( \frac{2n+1}{2} \frac{\pi x}{L} \right)
\]
Also
\[
u_t(x, t) = \sum_{n=0}^{\infty} \left[ B_n \left( \frac{2n+1}{2} \frac{\pi a}{L} \right) \cos \left( \left( \frac{2n+1}{2} \frac{\pi a t}{L} \right) \right) \right] \sin \left( \frac{2n+1}{2} \frac{\pi x}{L} \right)
\]
\[- \sum_{n=0}^{\infty} \left[ C_n \left( \frac{2n+1}{2} \frac{\pi a}{L} \right) \sin \left( \left( \frac{2n+1}{2} \frac{\pi a t}{L} \right) \right) \right] \sin \left( \frac{2n+1}{2} \frac{\pi x}{L} \right)
\]
Therefore
\[
u_t(x, 0) = g(x) = \sum_{n=0}^{\infty} B_n \left( \frac{2n+1}{2} \frac{\pi a}{L} \right) \sin \left( \frac{2n+1}{2} \frac{\pi x}{L} \right)
\]
11. \(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + u = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0\)
\(u(0, t) = u(L, t) = 0, \quad t > 0\)
\(u(x, 0) = f(x), \quad 0 < x < L\)
\[ \frac{\partial u}{\partial t} (x, 0) = g(x), \quad 0 < x < L \]

Assume the solution has the form \( u(x, t) = X(x)T(t) \).

Substituting this into the partial diff. eq. yields

\[
X(x)T''(t) + X(x)T'(t) + X(x)T(t) = a^2 X''(x)T(t)
\]

\[
\frac{T''(t) + T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}
\]

\[
K \Rightarrow T''(t) + T'(t) + (1 - a^2 K)T(t) = 0
\]

\[
\frac{X''(x)}{X(x)} = K \Rightarrow X''(x) - KX(x) = 0
\]

Substituting \( u(x, t) = X(x)T(t) \) into the boundary conditions yields

\[
X(0)T(t) = 0 = X(L)T(t), \quad t > 0
\]

\[
\Rightarrow X(0) = X(L) = 0
\]

\[
X''(x) - KX(x) = 0 \text{ with } X(0) = X(L) = 0
\]

\[
K = -(n\pi/L)^2, \quad n = 1, 2, 3, \ldots
\]

Now, plug \( K \) into the above equation

\[
T''(t) + T'(t) + (1 + \frac{a^2 n^2 \pi^2}{L^2})T(t) = 0, \quad n = 1, 2, 3, \ldots
\]

\[
r^2 + r + (1 + \frac{a^2 n^2 \pi^2}{L^2}) = 0
\]

\[
r = -\frac{1}{2} \pm \frac{\sqrt{3L^2 + 4a^2 n^2 \pi^2}}{2L}, \quad n = 1, 2, 3, \ldots
\]

\[
T_n(t) = e^{-t/2} \left[ B_n \cos \left( \frac{\sqrt{3L^2 + 4a^2 n^2 \pi^2}}{2L} t \right) + C_n \sin \left( \frac{\sqrt{3L^2 + 4a^2 n^2 \pi^2}}{2L} t \right) \right]
\]

Let \( \beta_n = \frac{\sqrt{3L^2 + 4a^2 n^2 \pi^2}}{2L} \) tp get

\[
T_n(t) = e^{-t/2} \left[ B_n \cos (\beta_n t) + C_n \sin (\beta_n t) \right]
\]

\[
u_n(x, t) = X_n(x)T_n(t) = A_n e^{-t/2} \left[ B_n \cos (\beta_n t) + C_n \sin (\beta_n t) \right] \sin (n\pi x/L)
\]

\[
u(x, t) = \sum_{n=1}^{\infty} \left[ a_n \cos (\beta_n t) + b_n \sin (\beta_n t) \right] \sin (n\pi x/L)
\]

where, \( a_n = A_n B_n \) and \( b_n = A_n C_n \)

Since

\[
\frac{\partial u(x, t)}{\partial t} = \sum_{n=1}^{\infty} \left\{ (-1/2)e^{-t/2}[a_n \cos (\beta_n t) + b_n \sin (\beta_n t)] + e^{-t/2}[-a_n \beta_n \sin (\beta_n t) + b_n \beta_n \cos (\beta_n t)] \right\} \sin (n\pi x/L)
\]

\[
\frac{\partial u(x, 0)}{\partial t} = 0 = \sum_{n=1}^{\infty} \left\{ -\frac{a_n}{2} + b_n \beta_n \right\} \sin (n\pi x/L)
\]

\[
\Rightarrow -\frac{a_n}{2} + b_n \beta_n = 0 \Rightarrow b_n = \frac{a_n}{2\beta_n}, \quad n = 1, 2, 3, \ldots
\]

Thus,

\[
u(x, t) = \sum_{n=1}^{\infty} a_n e^{-t/2} \left[ \cos (\beta_n t) + \frac{1}{2\beta_n} \sin (\beta_n t) \right] \sin (n\pi x/L)
\]

To find \( a_n \) use the remaining initial condition to obtain

\[
u(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \sin (n\pi x/L)
\]
Therefore,

\[ a_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) \, dx \]

Formal solution to the problem is given by

\[ u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\nu^2 t} [\cos(\beta_n t) + \frac{1}{2\beta_n} \sin(\beta_n t)] \sin(n\pi x/L) \]

where

\[ a_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) \, dx \]

and

\[ \beta_n = \frac{\sqrt{3L^2 + 4\alpha^2 n^2 \pi^2}}{2L} \]