

# MA 221 Homework Solutions

## Due date: April 1, 2014

8.3 p. 445 17, 19, 21, 25 27

17.) Find at least the first four nonzero terms in a power series expansion about  $x = 0$  for a general solution to a given differential equation.

$$w'' - x^2 w' + w = 0$$

$$\begin{aligned} w(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \\ &= \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

$$w = \sum_{n=0}^{\infty} a_n x^n$$

$$w' = \sum_{n=1}^{\infty} a_n(n) x^{n-1}$$

$$w'' = \sum_{n=2}^{\infty} a_n(n)(n-1) x^{n-2}$$

$$\begin{aligned} w'' - x^2 w' + w &= \sum_{n=0}^{\infty} a_n x^n - x^2 \sum_{n=1}^{\infty} a_n(n) x^{n-1} \\ &\quad + \sum_{n=2}^{\infty} a_n(n)(n-1) x^{n-2} = 0 \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_n(n) x^{n+1} \\ + \sum_{n=2}^{\infty} a_n(n)(n-1) x^{n-2} = 0 \end{aligned}$$

$$k = n - 2; n = k + 2; n = 2, k = 0; n - 1 = k + 1$$

$$\sum_{n=2}^{\infty} a_n(n)(n-1) x^{n-2} = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k$$

$$k = n + 1; n = k - 1; n = 1, k = 2$$

$$\sum_{n=1}^{\infty} a_n(n) x^{n+1} = \sum_{k=2}^{\infty} a_{k-1}(k-1) x^k$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_k x^k$$

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_n(n) x^{n+1} + \sum_{n=2}^{\infty} a_n(n)(n-1) x^{n-2} \\ = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k - \sum_{k=2}^{\infty} a_{k-1}(k-1) x^k + \sum_{k=0}^{\infty} a_k x^k \\ = 0 \end{aligned}$$

$$\begin{aligned} (2)(1)a_2 x^0 + (3)(2)a_3 x + \sum_{k=2}^{\infty} a_{k+2}(k+2)(k+1) x^k \\ - \sum_{k=2}^{\infty} a_{k-1}(k-1) x^k + a_0 x^0 + a_1 x + \sum_{k=2}^{\infty} a_k x^k = 0 \end{aligned}$$

$$a_0 + 2a_2 + (6a_3 + a_1)x + \sum_{k=2}^{\infty} [a_{k+2}(k+2)(k+1) - (k-1)a_{k-1} + a_k]x^k = 0$$

$$a_0 = a_0 \quad a_1 = a_1$$

$$a_0 + 2a_2 = 0 \quad 6a_3 + a_1 = 0$$

$$a_2 = -\frac{a_0}{2} \quad a_3 = -\frac{a_1}{6}$$

$$(k+2)(k+1)a_{k+2} - (k-1)a_{k-1} + a_k = 0$$

$$a_{k+2} = \frac{(k-1)a_{k-1} - a_k}{(k+2)(k+1)}$$

$$a_0 = a_0$$

$$a_1 = a_1$$

$$k = 0 : a_2 = -\frac{a_0}{2}$$

$$k = 1 : a_3 = -\frac{a_1}{6}$$

$$k = 2 : a_4 = \frac{(2-1)a_1 - a_2}{(2+1)(2+2)} = \frac{a_1 - a_2}{(4)(3)} = \frac{2a_1 + a_0}{24}$$

$$\begin{aligned} w(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = \\ &= a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 + \frac{2a_1 + a_0}{24}x^4 = \\ &= a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 + \frac{a_1}{12}x^4 + \frac{a_0}{24}x^4 = \\ &= a_0(1 - \frac{x^2}{2} + \frac{x^4}{24}) + a_1(x - \frac{x^3}{6} + \frac{x^4}{12}) \end{aligned}$$

19.) Find a power-series expansion about  $x = 0$  for a general solution of

$$y' - 2xy = 0$$

Your answer should include a general formula for the coefficients.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n(n) x^{n-1}$$

so the DE leads to

$$\sum_{n=1}^{\infty} a_n(n) x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Shifting the first sum by letting  $k+1 = n-1$  or  $n = k+2$  we get

$$\sum_{k=-1}^{\infty} a_{k+2}(k+2) x^{k+1} - 2 \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

or after replacing the place keepers by  $m$

$$a_1 + \sum_{m=0}^{\infty} [a_{m+2}(m+2) - 2a_m] x^{m+1} = 0$$

Thus

$$a_1 = 0$$

$$a_{m+2} = \frac{2}{m+2} a_m \quad m = 0, 1, 2, \dots$$

Substituting values for  $m$  we get

$$\begin{aligned} m = 0 &\Rightarrow a_2 = \frac{2}{2} a_0 \\ m = 1 &\Rightarrow a_3 = \frac{2}{3} a_1 = 0 \\ m = 2 &\Rightarrow a_4 = \frac{2}{4} a_2 = \frac{2^2 a_0}{2 \cdot 4} \\ m = 3 &\Rightarrow a_5 = 0 \\ m = 4 &\Rightarrow a_6 = \frac{2}{6} a_4 = \frac{2^3 a_0}{2 \cdot 4 \cdot 6} \end{aligned}$$

The pattern is therefore

$$a_{2p} = \frac{2^p}{2 \cdot 4 \cdot 6 \cdots (2p)} a_0 = \frac{2^p}{2^p (1 \cdot 2 \cdot 3 \cdots p)} a_0 = \frac{a_0}{p!}$$

and

$$y(x) = a_0 \sum_{p=0}^{\infty} \frac{x^{2p}}{p!}$$

$$21.) \quad y'' - xy' + 4y = 0$$

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} a_n(n)x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} \end{aligned}$$

Plugging into the DE, we have,

$$\begin{aligned} \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - x \sum_{n=1}^{\infty} a_n(n)x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \Rightarrow \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - \sum_{n=1}^{\infty} a_n(n)x^n + \sum_{n=0}^{\infty} 4a_n x^n &= 0 \end{aligned}$$

Now, shift the index of the first sum by letting  $k = n - 2$ , and we let  $k = n$  in the other two power series.

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} a_k(k)x^k + \sum_{k=0}^{\infty} 4a_k x^k = 0$$

Now,

$$(2)(1)a_2x^0 + \sum_{k=1}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} a_k(k)x^k + 4a_0x^0 + \sum_{k=1}^{\infty} 4a_k x^k = 0$$

$\Rightarrow$

$$2a_2 + 4a_0 + \sum_{k=1}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} a_k(k)x^k + \sum_{k=1}^{\infty} 4a_kx^k = 0$$

$\Rightarrow$

$$2a_2 + 4a_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} + (-k+4)a_k]x^k = 0$$

Now, setting each coefficient of the power series to zero, we see that,

$$2a_2 + 4a_0 \Rightarrow a_2 = \frac{-4a_0}{2} = -2a_0,$$

$$(k+2)(k+1)a_{k+2} + (-k+4)a_k = 0 \Rightarrow a_{k+2} = \frac{(k-4)a_k}{(k+2)(k+1)}, k \geq 1,$$

Thus we have:

$$k = 1 \Rightarrow a_3 = \frac{-3a_1}{(2)(3)} = -\frac{a_1}{2}$$

$$k = 2 \Rightarrow a_4 = \frac{-2a_2}{(4)(3)} = \frac{(-2)(-4)a_0}{(4)(3)(2)} = \frac{a_0}{3}$$

$$k = 3 \Rightarrow a_5 = \frac{-a_3}{(5)(4)} = \frac{(-3)(-1)a_1}{(5)(4)(3)(2)} = \frac{a_1}{40}$$

$$k = 4 \Rightarrow a_6 = 0$$

$$k = 5 \Rightarrow a_7 = \frac{a_5}{(7)(6)} = \frac{(-3)(-1)(1)(3)a_1}{(7)(6)(5)(4)(3)(2)} = \frac{a_1}{560}$$

$$k = 6 \Rightarrow a_8 = \frac{2a_6}{(8)(7)} = 0$$

$\Rightarrow$

$$a_{n+1} = \frac{(-3)(-1)(1)\dots(2n-5)a_1}{(2n+1)!}$$

Substituting these expressions for the coefficients into the solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$y(x) = a_0 + a_1 x - 2a_0 x^2 - \frac{a_1}{2} x^3 + \frac{a_0}{3} x^4 + \frac{a_1}{40} x^5 + \dots$$

$$+ \frac{(-3)(-1)(1)\dots(2n-5)a_1}{(2n+1)!} x^{2n+1} + \dots$$

$$= a_0[1 - 2x^2 + \frac{x^4}{3}] + a_1[x - \frac{x^3}{2} + \frac{x^5}{40}] + \frac{(-3)(-1)(1)\dots(2n-5)a_1}{(2n+1)!} x^{2n+1} + \dots$$

$$= a_0[1 - 2x^2 + \frac{x^4}{3}] + a_1[x + \sum_{k=1}^{\infty} \frac{(-3)(-1)(1)\dots(2n-5)a_1}{(2n+1)!} x^{2n+1}]$$

In problems 25 and 27 find at least the first four non zero terms in a power series expansion about  $x = 0$  for a general solution to the given the Initial Value Problem.

$$25.) \quad w'' + 3xw' - w = 0; \quad w(0) = 2, \quad w'(0) = 0$$

$$\begin{aligned} w &= \sum_{n=0}^{\infty} a_n x^n \\ w' &= \sum_{n=1}^{\infty} a_n(n) x^{n-1} \\ w'' &= \sum_{n=2}^{\infty} a_n(n)(n-1) x^{n-2} \end{aligned}$$

$$w(0) = a_0 = 2, \quad w'(0) = a_1 = 0$$

Plugging into the DE we have

$$\begin{aligned} \sum_{n=2}^{\infty} a_n(n)(n-1) x^{n-2} + 3x \sum_{n=1}^{\infty} a_n(n) x^{n-1} - \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=2}^{\infty} a_n(n)(n-1) x^{n-2} + \sum_{n=1}^{\infty} 3a_n(n) x^n - \sum_{n=0}^{\infty} a_n x^n &= 0 \end{aligned}$$

Now, shift the index of the first sum by letting  $k = n - 2$ , and shift the index of the second sum to  $k = n$  and we let  $k = n$  in the last power series.

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} 3a_k(k)x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

We can start all of the summations at the same point if we remove the first term in the first and last power series from above. We then have:

$$(2a_2 - a_0) + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} + 3ka_k - a_k] x^k = 0$$

By equating the coefficients we see all of the terms in the power series must be equal to zero. Then:

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$$

$$(k+2)(k+1)a_{k+2} + a_k(3k-1) = 0$$

$$a_{k+2} = \frac{-a_k(3k-1)}{(k+2)(k+1)} \Rightarrow k \geq 1$$

Thus we have,

$$\begin{aligned} k = 1 \Rightarrow a_3 &= \frac{-a_1(3-1)}{(1+2)(1+1)} = -\frac{2a_1}{6} = -\frac{a_1}{3} \\ k = 2 \Rightarrow a_4 &= \frac{-a_2(3(2)-1)}{(2+2)(2+1)} = -\frac{5a_2}{12} = -\frac{5\left(\frac{a_0}{2}\right)}{12} = -\frac{5a_0}{24} \\ k = 3 \Rightarrow a_5 &= \frac{-a_3(3(3)-1)}{(3+2)(3+1)} = \frac{8a_3}{20} = \frac{-8\left(-\frac{a_1}{3}\right)}{20} = \frac{2a_1}{15} \end{aligned}$$

Substituting these expressions for the coefficients into the solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

yields

$$\begin{aligned} y(x) &= a_0 + a_1 x + \frac{a_0}{2} x^2 - \frac{a_1}{3} x^3 - \frac{5a_0}{24} x^4 + \frac{2a_1}{15} x^5 + \dots \\ &= a_0[1 + \frac{x^2}{2} - \frac{5}{24}x^4 + \dots] + a_1[x - \frac{1}{3}x^3 + \frac{2}{15}x^5 \dots] \end{aligned}$$

$$w(0) = a_0 = 2$$

$$y'(0) = a_1 = 0$$

$$y(x) = 2 + x^2 - \frac{5}{12}x^4 + \frac{11}{72}x^6 + \dots$$

$$27.) \quad (x+1)y'' - y = 0; \quad y(0) = 0, \quad y'(0) = 1$$

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} a_n(n)x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} \end{aligned}$$

Plugging into the DE we have

$$\begin{aligned} (x+1) \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-1} - \sum_{n=0}^{\infty} a_n x^n &= 0 \end{aligned}$$

Now, shift the index of the first sum by letting  $k = n - 2$ , and shift the index of the second sum to  $k=n-1$  and we let  $k=n$  in the last power series.

$$\begin{aligned} \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} a_{k+1}(k+1)(k)x^k - \sum_{k=0}^{\infty} a_k x^k &= 0 \\ a_2(2)(1)x^0 + \sum_{k=1}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} a_k(k)(k-1)x^{k-1} - a_0x^0 - \sum_{k=0}^{\infty} a_k x^k &= 0 \end{aligned}$$

$\Rightarrow$

$$2a_2 - a_0 + \sum_{k=1}^{\infty} [a_{k+2}(k+2)(k+1) + a_{k+1}(k+1)(k) - a_k]x^k = 0$$

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$$

$$a_{k+2}(k+2)(k+1) + a_{k+1}(k+1)(k) - a_k = 0$$

$\Rightarrow$

$$a_{k+2} = \frac{a_k - a_{k+1}(k+1)(k)}{(k+2)(k+1)}, k \geq 1$$

Thus we have,

$$k = 1 \Rightarrow a_3 = \frac{a_1 - a_2(2)}{(3)(2)} = \frac{a_1 - a_0}{6}$$

$$\begin{aligned} k = 2 \Rightarrow a_4 &= \frac{a_2 - a_3(3)(2)}{(4)(3)} = \frac{\frac{a_0}{2} - a_1 + a_0}{12} \\ &= \frac{3\frac{a_0}{2} - a_1}{12} = \frac{3a_0 - 2a_1}{24} \end{aligned}$$

$$\begin{aligned} k = 3 \Rightarrow a_5 &= \frac{a_3 - a_4(4)(3)}{(5)(4)} = \frac{\frac{a_1 - a_0}{6} - \frac{3}{2}a_0 + a_1}{120} \\ &= \frac{a_1 - a_0 - 9a_0 + 6a_1}{120} = \frac{7a_1 - 10a_0}{120} \end{aligned}$$

Substituting these expressions for the coefficients into the solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

yields

$$y(x) = a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1 - a_0}{6} x^3 + \frac{3a_0 - 2a_1}{24} x^4 + \frac{7a_1 - 10a_0}{120} x^5 + \dots$$

$$y(0) = a_0(1) = 0 \Rightarrow a_0 = 0$$

$$y'(0) = a_1 = 1$$

$$y(x) = x + \frac{1}{6} x^3 - \frac{1}{12} x^4 + \frac{7}{120} x^5 + \dots$$

SNB check  $(x+1)y'' - y = 0$ , Series solution is:

$$\left\{ y(0) + xy'(0) + \frac{1}{2}x^2y(0) - x^3 \left( \frac{1}{6}y(0) - \frac{1}{6}y'(0) \right) + x^4 \left( \frac{1}{8}y(0) - \frac{1}{12}y'(0) \right) - x^5 \left( \frac{1}{12}y(0) - \frac{7}{120}y'(0) \right) \right.$$