MA 221 Homework Solutions
Due date: April 3, 2014

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Problems 1, 3, 5, 6, 7, 13
(Underlined problems are to be handed in)
In problems 1, 3, 5 and 6 determine the solutions, if any, to the given boundary problem.
1.) $y'' + 2y' + 26y = 0$
$y(0) = 1, y(\pi) = -e^{-\pi}$

$$r^2 + 2r + 26 = 0$$
$$r = -1 \pm 5i$$

Therefore

$$y(x) = c_1 e^{-x} \cos 5x + c_2 e^{-x} \sin 5x$$

The BCs imply

$$y(0) = c_1 = 1$$
$$y(\pi) = c_1 e^{-\pi} \cos 5\pi = -e^{-\pi}$$
$$-c_1 e^{-\pi} = -e^{-\pi}$$
$$c_1 = 1$$

Thus

$$y(x) = e^{-x} \cos 5x + c_2 e^{-x} \sin 5x$$

Where C_2 is arbitrary

3.) $y'' - 4y' + 13y = 0$
$y(0) = 0, y(\pi) = 0$

$$r^2 - 4r + 13 = 0$$
$$r = 2 \pm 3i$$

Therefore

$$y(x) = c_1 e^{2t} \sin 3t + c_2 e^{2t} \cos 3t$$

The BCs imply

$$y(0) = c_2 = 0$$
$$y(\pi) = c_1 e^{2\pi} \sin 3\pi = 0$$

For all c_1. Thus

$$y = c_1 e^{2t} \sin 3t$$

5.) $y'' + y = \sin 2x \qquad y(0) = y(2\pi) \qquad y'(0) = y'(2\pi)$

$$r^2 + 1 = 0$$
$$r = \pm i$$
$$\alpha = 0, \beta = 1$$
$$y_h(t) = c_1 e^0 \cos x + c_2 e^0 \sin x$$
$$y_h(t) = c_1 \cos x + c_2 \sin x$$
$$y_p = A \sin 2x + B \cos 2x$$
$$y_p' = 2A \cos 2x - 2B \sin 2x$$
$$y_p'' = -4A \sin 2x - 4B \cos 2x$$
$$-4A \sin 2x - 4B \cos 2x + A \sin 2x + B \cos 2x = \sin 2x$$
$$-3A \sin 2x - 3B \cos 2x = \sin 2x$$
$$-3A = 1, -3B = 0$$
$$A = -1/3, B = 0$$
$$y_p = (-1/3) \sin 2x$$
$$y(x) = c_1 \cos x + c_2 \sin x - (1/3) \sin 2x$$

where c_1 and c_2 are the real numbers

$y(0) = c_1$ and $y(2\pi) = c_1$ so the condition $y(0) = y(\pi)$ implies nothing.

$$y'(x) = -c_1 \sin x + c_2 \cos x - \frac{2}{3} \cos 2x$$

Thus

$y'(0) = c_2 - \frac{2}{3}$ and $y'(2\pi) = c_2 - \frac{2}{3}$ so the condition $y'(0) = y'(2\pi)$ also implies nothing.

Thus

$$y(x) = c_1 \cos x + c_2 \sin x - (1/3) \sin 2x$$

6.) $y'' - y = x$

$y(0) = 3, y'(1) = 2e - e^{-1} - 1$

$$r^2 - 1 = 0$$
$$r = \pm 1$$
$$y_h = c_1 e^x + c_2 e^{-x}$$
$$y_p = Ax + B$$
$$y_p' = A$$
$$y_p'' = 0$$
$$0 - Ax - B = x$$
$$A = -1, B = 0$$
$$y_p = -x$$

Substituting

$$y(0) = 3, y'(1) = 2e - e^{-1} - 1$$
$$c_1 = 2, c_2 = 1$$
$$y = 2e^x + e^{-x} - x$$

7)

$$y'' + y = e^x \quad y(0) = 0 \quad y(\pi) + y'(\pi) = 0$$

For this equation

$$p(r) = r^2 + 1$$

which has root $\pm i$ so

$$y_h = c_1 \cos x + c_2 \sin x$$
$$y_p = \frac{e^x}{p(1)} = \frac{e^x}{2}$$

so

$$y(x) = c_1 \cos x + c_2 \sin x + \frac{e^x}{2}$$

$$y(0) = c_1 + \frac{1}{2} \quad \Rightarrow c_1 = -\frac{1}{2}$$

and

$$y(x) = -\frac{1}{2} \cos x + c_2 \sin x + \frac{e^x}{2}$$
$$y'(x) = \frac{1}{2} \sin x + c_2 \cos x + \frac{e^x}{2}$$

$$y(\pi) + y'(\pi) = \frac{1}{2} + \frac{e^\pi}{2} - c_2 + \frac{e^\pi}{2} = 0$$

Thus

$$c_2 = \frac{1}{2} + e^\pi$$

and

$$y(x) = -\frac{1}{2} \cos x + \left(\frac{1}{2} + e^\pi\right) \sin x + \frac{e^x}{2}$$

In Problem 13 find all the real eigenvalues and eigenfunctions for the given eigenvalue problem.

13.) $y'' + \lambda y = 0;$

$y(0) = 0, \qquad y'(1) = 0$

The auxiliary equation for this problem is: $r^2 + \lambda = 0.$

To find eigenvalues that yield nontrivial solutions we will consider the three cases

$\lambda < 0$

$\lambda = 0$

$\lambda > 0$

Case 1: $\lambda < 0$ Let $\lambda = -\alpha^2,$ where $\alpha \neq 0.$ The DE becomes

$$y'' - \alpha^2 y = 0$$

In this case, the roots to the auxiliary equation are $\pm\alpha$ Therefore, a general solution to the differential equation is given by:

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

By applying the BC's:
$y(0) = c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$
Thus

$$y(x) = c_1(e^{\alpha x} - e^{-\alpha x})$$

In order to apply the second BC, we need to find $y'(x)$. Thus we have:

$$y'(x) = c_1 \alpha (e^{\alpha x} + e^{-\alpha x})$$

Plugging in the second BC $y'(1) = 0$

$$y'(1) = c_1 \alpha (e^{\alpha} + e^{-\alpha}) = 0$$

Since $e^{\alpha} + e^{\alpha} \neq 0$, the only way the equation above can be true is for $c_1 = 0$. So in this case we have only the trivial solution. Thus, there are no eigenvalues for $\lambda < 0$.

Case 2: $\lambda = 0$
In this case we are solving the differential equation $y'' = 0$. This equation has a general solution given by:
$y(x) = c_1 + c_2 x \quad \Rightarrow \quad y'(x) = c_2$
By applying the boundary conditions, we obtain
$y(0) = c_1 = 0;$
$y'(1) = c_2 = 0$
Thus, $c_1 = c_2 = 0$, and zero is not an eigenvalue

Case 3: $\lambda > 0$ Let $\lambda = \beta^2$ where $\beta \neq 0.$ The DE becomes

$$y'' + \beta^2 y = 0$$

In this case the roots to the associated auxiliary equation are $r = \pm\beta i$
Therefore, the general solution is given by

$$y(x) = c_1 \cos\beta x + c_2 \sin\beta x$$

By applying the first boundary condition, we obtain
$y(0) = c_1 = 0 \quad \Rightarrow$

$$y(x) = c_2 \sin\beta x$$

In order to apply the second BC we need to find $y'(x)$. Thus,

$$y'(x) = c_2 \beta \cos\beta x$$

Pluging in the BC

$$y'(1) = c_2 \beta \cos\beta = 0$$

Therefore, in order to obtain a solution other that the trivial solution, we must have
$\cos\beta = 0 \quad \Rightarrow \quad \beta = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, 1, 2\ldots$

$\Rightarrow \lambda_n = \beta^2 = \left(n + \frac{1}{2}\right)^2 \pi^2, \qquad$ with $n = 0, 1, 2\ldots$

For these eigenvalues λ_n, we have the corresponding eigenfunctions,

$$y_n(x) = c_n \sin\left[\left(n + \frac{1}{2}\right)\pi x\right] \qquad \text{with } n = 0, 1, 2.\ldots.$$

where c_n is an arbitrary nonzero constant.