MA 221 Homework Solutions
Due date: April 8, 2014

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Problems <u>15</u>, <u>17</u> and <u>19</u>
(Underlined problems are to be handed in)

In Problems 15, 17 and 19 find all the real eigenvalues and eigenfunctions for the given eigenvalue problem.
13.) $y'' + \lambda y = 0;$
$y(0) = 0, \qquad y'(1) = 0$
The auxiliary equation for this problem is: $r^2 + \lambda = 0.$
To find eigenvalues that yield nontrivial solutions we will consider the three cases
$\lambda < 0$
$\lambda = 0$
$\lambda > 0$

Case 1: $\lambda < 0$ Let $\lambda = -\alpha^2,$ where $\alpha \neq 0.$ The DE becomes

$$y'' - \alpha^2 y = 0$$

In this case, the roots to the auxiliary equation are $\pm\alpha$ Therefore, a general solution to the differential equation is given by:

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

By applying the BC's:
$y(0) = c_1 + c_2 = 0 \qquad \Rightarrow \qquad c_2 = -c_1$
Thus

$$y(x) = c_1(e^{\alpha x} - e^{-\alpha x})$$

In order to apply the second BC, we need to find $y'(x)$. Thus we have:

$$y'(x) = c_1\alpha(e^{\alpha x} + e^{-\alpha x})$$

Plugging in the second BC $y'(1) = 0$

$$y'(1) = c_1\alpha(e^{\alpha} + e^{-\alpha}) = 0$$

Since $e^{\alpha} + e^{\alpha} \neq 0$, the only way the equation above can be true is for $c_1 = 0$. So in this case we have only the trivial solution. Thus, there are no eigenvalues for $\lambda < 0$.

Case2: $\lambda = 0$
In this case we are solving the differential equation $y'' = 0$. This equation has a general solution given by:
$y(x) = c_1 + c_2 x \qquad \Rightarrow \qquad y'(x) = c_2$
By applying the boundary conditions, we obtain
$y(0) = c_1 = 0;$
$y'(1) = c_2 = 0$

Thus, $c_1 = c_2 = 0$, and zero is not an eigenvalue

Case 3: $\lambda > 0$ Let $\lambda = \beta^2$ where $\beta \neq 0.$ The DE becomes

$$y'' + \beta^2 y = 0$$

In this case the roots to the associated auxiliary equation are $r = \pm\beta i$

Therefore, the general solution is given by

$$y(x) = c_1 \cos \beta x + c_2 \sin \beta x$$

By applying the first boundary condition, we obtain

$y(0) = c_1 = 0 \qquad \Rightarrow$

$$y(x) = c_2 \sin \beta x$$

In order to apply the second BC we need to find $y'(x)$. Thus,

$$y'(x) = c_2 \beta \cos \beta x$$

Pluging in the BC

$$y'(1) = c_2 \beta \cos \beta = 0$$

Therefore, in order to obtain a solution other that the trivial solution, we must have

$\cos \beta = 0 \qquad \Rightarrow \qquad \beta = \left(n + \frac{1}{2}\right)\pi, \qquad n = 0, 1, 2\ldots$

$\Rightarrow \lambda_n = \beta^2 = \left(n + \frac{1}{2}\right)^2 \pi^2, \qquad$ with $n = 0, 1, 2\ldots$

For these eigenvalues λ_n, we have the corresponding eigenfunctions,

$$y_n(x) = c_n \sin\left[\left(n + \frac{1}{2}\right)\pi x\right] \qquad \text{with } n = 0, 1, 2.\ldots.$$

where c_n is an arbitrary nonzero constant.

15.) $y'' + 3y + \lambda y = 0;$

$y'(0) = 0, \qquad y'(\pi) = 0$

The auxiliary equation for this problem is: $r^2 + (\lambda + 3) = 0$ and $r = \pm\sqrt{-(3 + \lambda)}$

To find the eigenvalues which yield nontrivial solutions, three cases must be considered:

$\lambda + 3 < 0$

$\lambda + 3 = 0$

$\lambda + 3 > 0$

Case1: $\lambda + 3 < 0$ Let $\lambda + 3 = -\alpha^2$ where $\alpha \neq 0$

In this case the roots to the auxiliary equation are the real numbers $\pm\sqrt{-(\lambda + 3)}$

The general solution to $y'' - \alpha^2 y = 0$ is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

By applying the first boundary condition, we obtain:

$y'(x) = c_1 \alpha e^{\alpha x} - c_2 \alpha e^{-\alpha x} = \alpha(c_1 e^{\sqrt{-(\lambda+3)}\,x} - c_2 e^{-\sqrt{-(\lambda+3)}\,x})$

$y'(0) = \alpha(c_1 - c_2) = 0 \Rightarrow c_1 - c_2 = 0 \Rightarrow c_1 = c_2$

$y'(\pi) = \alpha(c_1 e^{\alpha\pi} - c_1 e^{-\alpha\pi}) = \alpha c_1(e^{\alpha\pi} - e^{-\alpha\pi}) = 0;$ since $\alpha \neq 0$ and $e^{\alpha\pi} - e^{-\alpha\pi} \neq 0 \Rightarrow c_1 = c_2 = 0$

In this case, we have only the trivial solution. There are no eigenvalues for $\lambda + 3 < 0$.

Case 2: $\lambda + 3 = 0$

In this case we are solving the differential equation $y'' = 0$. This equation has a general solution given by:

$y(x) = c_1 + c_2 x$
$y'(x) = c_2$
By applying the boundary conditions, we obtain
$y'(0) = c_2 = 0;$
$y'(\pi) = c_2 = 0$
Thus, c_1 is arbitary and zero is an eigenvalue with eigenfunction $y(x) = C,\ C$ any constant.
Case 3:$\lambda + 3 > 0$ Let $\lambda + 3 = \beta^2$ where $\beta \neq 0$
The DE becomes

$$y'' + \beta^2 y = 0$$

Therefore, the general solution is given by

$$y(x) = c_1 \cos\beta x + c_2 \sin\beta x$$

By applying the first boundary condition, we obtain
$y'(x) = \beta(-c_1 \sin\beta x + c_2 \cos\beta x)$
$y'(0) = \beta c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow y'(x) = \beta(-c_1 \sin\beta x)$
$y'(\pi) = \beta(-c_1 \sin(\beta\pi) = 0,$ Since $\beta \neq 0$ and we want
$c_1 \neq 0 \Rightarrow \beta = n \Rightarrow \lambda + 3 = \beta^2 = n^2 \Rightarrow \lambda = n^2 - 3$
$\Rightarrow \lambda_n = n^2 - 3$ with $n = 0, 1, 2\ldots$
For these eigenvalues λ_n, we have the corresponding eigenfunctions,
$y_n(x) = c_n \cos nx$ with $n = 0, 1, 2.\ldots.$ where c_n is an arbitrary nonzero constant.

17.) $y'' + \lambda y = 0$ $2y(0) + y'(0) = 0$ $y(\pi) = 0$
The find the eigenvalues which yield nontrivial solutions, three cases must be considered:
$\lambda < 0$
$\lambda = 0$
$\lambda > 0$

Case 1: $\lambda < 0$
$\lambda = -\mu^2$ $\mu > 0$ and the DE is $y'' - \mu^2 y = 0$

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

For convenience we introduce the hyperbolic sine and cosine

$$\cosh(\mu x) = \frac{e^{\mu x} + e^{-\mu x}}{2}$$

$$\sinh(\mu x) = \frac{e^{\mu x} - e^{-\mu x}}{2}$$

and write the solution above in terms of these functions. Then

$$\begin{aligned} y(x) &= c_1(\cosh(\mu x) + \sinh(\mu x)) + c_2(\cosh(\mu x) - \sinh(\mu x)) \\ &= (c_1 + c_2)\cosh(\mu x) + (c_1 - c_2)\sinh(\mu x) \end{aligned}$$

Let

$$k_1 = (c_1 + c_2) \qquad k_2 = (c_1 - c_2)$$

then:

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} = k_1 \cosh(\mu x) + k_2 \sinh(\mu x)$$

$$y'(x) = \mu k_1 \sinh(\mu x) + \mu k_2 \cosh(\mu x)$$

$$2y(0) + y'(0) = 2k_1 + \mu k_2 = 0 \qquad \Rightarrow k_2 = \frac{-2k_1}{\mu}$$

so

$$y(x) = k_1\left(\cosh(\mu x) - \left(\frac{2}{\mu}\right)\sinh(\mu x)\right)$$

$$y(\pi) = k_1\left(\cosh(\mu\pi) - \left(\frac{2}{\mu}\right)\sinh(\mu\pi)\right) = 0$$

Since we want $k_1 \neq 0.$ then we must have

$$\frac{\mu}{2} = \tanh(\mu\pi)$$

so

$$\mu = 2\tanh(\mu\pi) \qquad \Rightarrow \qquad \lambda = -\mu^2 = -4\tanh^2(\mu\pi)$$

and

$$y(x) = k_1\left(\cosh(\mu x) - \left(\frac{2}{\mu}\right)\sinh(\mu x)\right)$$

Case 2: $\lambda = 0$
The DE becomes $y''(x) = 0,$ so $y(x) = ax + b \qquad y'(x) = a$
$2y(0) + y'(0) = 2b + a = 0 \qquad y(\pi) = a\pi + b = 0$
Thus $a = b = 0$ and we have only the trivial solution.

Case 3: $\lambda > 0$ Let $\lambda = \mu^2,$ where $\mu \neq 0.$ Then the DE becomes $y'' + \mu^2 y = 0$ and

$$y(x) = c_1 \sin \mu x + c_2 \cos \mu x$$
$$y'(x) = c_1 \mu \cos \mu x - c_2 \mu \sin x$$

$$2y(0) + y'(0) = 2c_2 + \mu c_1 = 0$$

$$y(\pi) = c_1 \sin \mu\pi + c_2 \cos \mu\pi = 0$$

Then:

$$c_2 = \frac{-\mu}{2} c_1$$

and

$$c_1\left(\sin \mu\pi - \frac{\mu}{2}\cos \mu\pi\right) = 0$$

so

$$\tanh \mu\pi = \frac{\mu}{2} \Rightarrow \mu = 2\tan \mu\pi$$

and

$$y = c\left(\sin \mu x - \frac{\mu}{2}\cos \mu x\right)$$

19.) $(xy')' + \lambda x^{-1} = 0 \qquad y'(0) = 0 \qquad y(e^{\pi}) = 0$
By the Cauchy-Euler equation
$(xy')' + \lambda x^{-1} = x^2 y'' + xy' + \lambda y = 0 \qquad x > 0$

Substituting $y = x^r$ gives $r^2 + \lambda = 0$ as the auxiliary equation for $x^2 y'' + xy' + \lambda y = 0$
Case 1: $\lambda < 0$: Let $\lambda = -\mu^2$ for $\mu > 0.$ The roots are $r = \pm\mu$
The general solution is: $y(x) = c_1 x^{\mu} + c_2 x^{-\mu}$
and $y'(x) = c_1 \mu x^{\mu} - c_2 \mu x^{-\mu-1} = \mu(c_1 x^{\mu} - c_2 x^{-\mu-1})$
Substituting into the first boundary condition gives
$y'(1) = \mu(c_1 - c_2) = 0$
Since $\mu > 0$
$c_1 - c_2 = 0 \quad \Rightarrow \quad c_1 = c_2 \quad \Rightarrow y(x) = c_1(x^{\mu} + x^{-\mu})$
Substituting this into the second condition yields:
$y(e^{\pi}) = c_1(e^{\mu\pi} + e^{-\mu\pi}) = 0$
Since $e^{\mu\pi} + e^{-\mu\pi} \neq 0$ the only way equation $c_1(e^{\mu\pi} + e^{-\mu\pi}) = 0$ can be true is for $c_1 = 0.$
In this case, we only have trivial solutions.
Case 2: $\lambda = 0$
In this case we are solving the differential equation $(xy')' = 0.$ This equation can be solved as follows:
$xy' = c_1 \quad \Rightarrow y' = \frac{c_1}{x} \quad \Rightarrow y(x) = c_2 + c_1 \ln x$
By applying the boundary conditions, we obtain
$y'(1) = c_1 = 0 \quad y(e^{\pi}) = c_2 + c_1 \ln(e^{\pi}) = c_2 + c_1\pi = 0$
Solving these equations simultaneously yields $c_1 = c_2 = 0.$ This, we again find only the trivial solution. Therefore, $\lambda = 0$ is not an eigenvalue.
Case 3: $\lambda > 0$
Let $\lambda = \mu^2$ for $\mu > 0.$ The roots of the auxiliary equation are $r \pm \mu i$
The general solution is:
$y(x) = c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)$
$y'(x) = -c_1(\frac{\mu}{x})\sin(\mu \ln x) + c_2(\frac{\mu}{x})\cos(\mu \ln x)$
By applying the first boundary condition, we obtain
$y'(1) = c_2\mu = 0 \quad c_2 = 0$
Applying the second boundary condition, we obtain
$y(e^{\pi}) = c_1 \cos(\mu \ln(e^{\pi})) = c_1 \cos(\mu\pi) = 0$
Therefore, in order to obtain a solution other than the trivial solution, we must have
$\cos(\mu\pi) = 0 \quad \Rightarrow \quad \mu\pi = (n + \frac{1}{2})\pi \quad n = 0, 1, 2.......$
$\Rightarrow \mu = n + \frac{1}{2} \quad \Rightarrow \lambda_n = (n + \frac{1}{2})^2 \quad n = 0, 1, 2.......$
Corresponding to the eigenvalues, λ_n's, we have the eigenfunctions:
$y_n(x) = c_n \cos\left[(n + \frac{1}{2})\ln x\right] \quad n = 0, 1, 2.....$
Where c_n is an arbitrary nonzero constant.