In problems 5 and 7 compute the Fourier sine series for the given function.

5) \( f(x) = -1 \) on \((0, 1)\) so \( L = 1 \)

If \( f(x) \) is a function defined on \([0, L]\), then its Fourier sine expansion is given by

\[
f(x) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{L} \right) \quad \text{where} \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) dx
\]

Here we have

\[
f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad \text{and} \quad a_n = \frac{2}{L} \int_{0}^{1} (\sin(n\pi x))dx
\]

\[
a_n = \frac{2}{L} \int_{0}^{1} (-1) \sin(n\pi x)dx = \frac{2}{n\pi} \cos(n\pi) \bigg|_{0}^{1}
\]

\[
= \frac{2}{n\pi} (\cos n\pi - \cos 0) = \frac{2\left((-1)^n - 1\right)}{n\pi}
\]

\( a_{2k} = 0 \) if \( n = 2k \) is even

\( a_{2k-1} = \frac{2\left((-1)^{2k-1} - 1\right)}{(2k-1)\pi} \) if \( n = 2k - 1 \) is odd

\( a_{2k} = 0 \) if \( n = 2k \) is even

\[
a_{2k-1} = -\frac{4}{(2k-1)\pi}
\]

Hence

\[
f(x) = -\frac{4}{\pi} \sum_{k=1}^{\infty} \left( \frac{1}{2k-1} \right) \sin(2k-1)\pi x
\]

7) \( f(x) = x^2 \) \( 0 < x < \pi \)

Here \( L = \pi \) so
\[
f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)
\]
where \( a_n = \frac{2}{\pi} \int_0^\pi x^2 \sin(nx)dx \)

To compute this integral we integrate by parts twice

\[
\left(\frac{\pi}{2}\right) a_n = \int_0^\pi x^2 \sin(nx)dx
\]

\[
= -x^2 \frac{\cos nx}{n} \bigg|_0^\pi + \frac{2}{n} \int_0^\pi x \cos nx dx
\]

\[
= -\frac{\pi^2 \cos n\pi}{n} + 0 + \frac{2}{n} \left[ x \frac{\sin nx}{n} \bigg|_0^\pi - \frac{1}{n} \int_0^\pi \sin nx dx \right]
\]

\[
= -\frac{\pi^2 \cos n\pi}{n} + \frac{2}{n} \left[ 0 - \frac{1}{n} \left( -x \cos nx \right) \bigg|_0^\pi \right]
\]

\[
= -\frac{\pi^2 \cos n\pi}{n} + \frac{2}{n^3} (\cos n\pi - 1) \quad n = 1.2.3.\ldots
\]

Since \( \cos n\pi = 1 \) if \( n \) is even and \( -1 \) if \( n \) is odd

\[
\left(\frac{\pi}{2}\right) a_n = -\frac{\pi^2 (-1)^n}{n} + \frac{2}{n^3} [(-1)^n - 1] \quad n = 1.2.3.\ldots
\]

Thus

\[
a_n = \frac{2\pi(-1)^{n+1}}{n} + \frac{4[(-1)^n - 1]}{\pi n^3}
\]

Therefore the Fourier sine series for \( f(x) = x^2 \) on \( (0,\pi) \) is given by

\[
\sum_{n=1}^{\infty} \left\{ \frac{2\pi(-1)^{n+1}}{n} + \frac{4[(-1)^n - 1]}{\pi n^3} \right\} \sin nx
\]

13) Find the Fourier cosine series for \( f(x) = e^x \) on \( 0 < x < 1 \).

Here we use the formulas given in the book. Namely

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{T} \right) \quad a_n = \frac{2}{T} \int_0^T f(x) \cos \left( \frac{n\pi x}{T} \right) dx
\]

Since \( T = 1 \) we have

\[
a_n = 2 \int_0^1 e^x \cos(n\pi x) dx
\]

Hence

\[
a_0 = 2 \int_0^1 e^x dx = 2(e - 1)
\]

From a table of integrals we have

\[
\int e^x \cos(n\pi x) dx = \frac{1}{\pi^2 n^2 + 1} (e^x \cos nx + \pi ne^x \sin nx)
\]
For \( n \geq 1 \) we have
\[
a_n = \frac{2}{\pi^2 n^2 + 1} \left( e^x \cos \pi nx + \pi ne^x \sin \pi nx \right) \bigg|_0^1
\]
\[
= \frac{2e \cos n\pi}{\pi^2 n^2 + 1} - \frac{2(1)}{\pi^2 n^2 + 1} = \frac{2(-1)^n e - 1}{\pi^2 n^2 + 1}
\]
Thus the Fourier cosine series for \( e^x \) on \( 0 < x < 1 \) is
\[
e - 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n e - 1}{\pi^2 n^2 + 1} \cos n\pi x
\]