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Ma 221 **Final Exam Solutions** **12/12/02**

Name: _________________________________ **ID**: __________________________

Lecture Section: _________

I pledge my honor that I have abided by the Stevens Honor System.

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This exam is in two parts. On Part I you are to answer all 7 questions. In Part II you are to choose *one* of the two questions and answer it. The point value of each problem is indicated. If you need more work space, continue the problem you are doing on the **other side of the page it is on**. You may *not* use a calculator on this exam. You must show how you obtained any result that you give. Be sure that you do all 8 problems.

A table of Laplace transforms and a table of integrals is given on the last page of the exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

#9 ________

Total Score ________

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Part I: **Answer all questions**

1. Solve

(a) (8 pts)

$$y' = y^2 e^{-x} \quad y(1) = 4$$

Solution: This equation is separable. Thus

$$\frac{dy}{y^2} = e^{-x} dx$$

Hence

$$-\frac{1}{y} = -e^{-x} + C$$

The initial condition implies

$$-\frac{1}{4} = -e^{-1} + C$$

so

$$C = e^{-1} - \frac{1}{4}$$

Thus

$$-\frac{1}{y} = -e^{-x} + e^{-1} - \frac{1}{4}$$

or

$$y = \frac{1}{e^{-x} - e^{-1} + \frac{1}{4}}$$

(b) (10 pts)

$$y'' - 4y' + 53y = 0 \quad y(0) = 7 \quad y'(0) = 14$$

Solution: The characteristic equation is

$$r^2 - 4r + 53 = 0$$

so

$$r = \frac{+4 \pm \sqrt{16 - 4(53)}}{2} = 2 \pm 7i$$

Thus

$$y(x) = C_1 e^{2x} \cos 7x + C_2 e^{2x} \sin 7x$$

To satisfy the initial conditions we have

$$y(0) = C_1 = 7$$

so

$$y(x) = 7e^{2x} \cos 7x + C_2 e^{2x} \sin 7x$$

Then

$$y'(x) = 14e^{2x} \cos 7x - 7e^{2x} \sin 7x + 2C_2 e^{2x} \sin 7x + 7C_2 e^{2x} \cos 7x$$

$$y'(0) = 14 + 7C_2 = 14 \Rightarrow C_2 = 0$$

Therefore

$$y(x) = 7e^{2x} \cos 7x$$

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(c) (7 pts.)

$$y' = 3x^2 - \frac{y}{x}$$

Solution: We write the equation as

$$y' + \frac{1}{x}y = 3x^2$$

This is a first order linear equation. The integrating factor is $e^{\int \frac{1}{x}dx} = x$. Multiplying the DE by x we have

$$xy' + y = (xy)' = 3x^3$$

so

$$y = \frac{3}{4}x^3 + \frac{C}{x}$$

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2. (a) (15 pts.) Solve

$$y'' + 2y' - 3y = 3x^2 - \frac{14}{3} + 8e^x$$

Solution: First we find the homogeneous solution. The characteristic equation is

$$p(r) = r^2 + 2r - 3 = (r+3)(r-1) = 0 \Rightarrow r = -3, 1$$

Therefore

$$y_h = C_1 e^x + C_2 e^{-3x}$$

We have to find a particular solution for $3x^2 - \frac{14}{3}$ and $8e^x$. Since e^x is a homogeneous solution, but xe^x is not, then

$$y_{p_1} = \frac{8xe^x}{p'(1)} = \frac{8xe^x}{4} = 2xe^x$$

For the polynomial we assume

$$y_{p_2}(x) = Ax^2 + Bx + C$$
$$y'_{p_2}(x) = 2Ax + B$$
$$y''_{p_2}(x) = 2A$$

The DE implies

$$2A + 4Ax + 2B - 3Ax^2 - 3Bx - 3C = 3x^2 - \frac{14}{3}$$

Hence

$$A = -1$$
$$4A - 3B = 0 \Rightarrow B = -\frac{4}{3}$$
$$2A + 2B - 3C = -\frac{14}{3} \Rightarrow -3C = 2 + \frac{8}{3} - \frac{14}{3} = 0 \Rightarrow C = 0$$

$$y_{p_2} = -x^2 - \frac{4}{3}x$$

Thus

$$y_g(x) = C_1 e^x + C_2 e^{-3x} + 2xe^x - x^2 - \frac{4}{3}x$$

SNB check: $y'' + 2y' - 3y = 3x^2 - \frac{14}{3} + 8e^x$, Exact solution is:
$y(x) = -\frac{1}{6}e^x\left(6e^{-x}x^2 + 8xe^{-x} - 12x + 3\right) + C_1 e^x + C_2 e^{-3x}$

2(b) (10 pts.) Find a general solution of

$$y'' + y = \tan x$$

Solution: We use Variation of Parameters.

$$y_h = C_1 \sin x + C_2 \cos x$$

so

$$y_p = v_1 \sin x + v_2 \cos x$$

The equations for v_1' and v_2' are

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$$v_1' \sin x + v_2' \cos x = 0$$
$$v_1' \cos x - v_2' \sin x = \tan x$$

$$v_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\sin x}{-1} = \sin x$$

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = -\tan x \sin x$$

$$v_1 = -\cos x$$

$$v_2 = -\int \tan x \sin x dx = \sin x - \ln(\sec x + \tan x)$$

Therefore

$$y_p = -\cos x \sin x + \cos x \sin x - \ln(\sec x + \tan x)$$

$$y_g = C_1 \sin x + C_2 \cos x - \ln(\sec x + \tan x)$$

SNB check $y'' + y = \tan x$, Exact solution is:
$y(x) = -\cos x \sin x + (\sin x - \ln(\sec x + \tan x)) \cos x + C_1 \sin x + C_2 \cos x$

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3. (a) (10 pts) Find

$$\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+4s+29}\right\}$$

Solution:

$$\frac{3s+2}{s^2+4s+29} = \frac{3s+2}{(s+2)^2+25} = \frac{3(s+2)-4}{(s+2)^2+25}$$
$$= 3\frac{s+2}{(s+2)^2+25} - \frac{4}{5}\frac{5}{(s+2)^2+25}$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{3s+4}{s^2+4s+29}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+25}\right\} - \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^2+25}\right\}$$

Using the shift theorem and the table at the end of the exam we have

$$\mathcal{L}^{-1}\left\{\frac{3s+4}{s^2+4s+29}\right\} = 3e^{-2t}\sin 5t - \frac{4}{5}e^{-2t}\cos 5t$$

.

(b) (15 pts) Use Laplace Transforms to solve:

$$y'' - 2y' - 3y = e^t, \quad y(0) = 1, \ y'(0) = 0$$

Solution:

$$\mathcal{L}\{y'' - 2y' - 3y\} = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

Thus

$$(s^2-2s-3)\mathcal{L}\{y\} - y(0)(s-2) - y'(0) = \frac{1}{s+1}$$
$$(s^2-2s-3)\mathcal{L}\{y\} = s-2+\frac{1}{s-1} = \frac{s^2-3s+3}{s-1}$$

:

$$\mathcal{L}\{y\} = \frac{s^2-3s+3}{(s-1)(s-3)(s+1)}$$

$$\frac{s^2-3s+3}{(s-1)(s-3)(s+1)} = \frac{A}{s-1}+\frac{B}{s-3}+\frac{C}{s+1} = -\frac{1}{4(s-1)}+\frac{3}{8(s-3)}+\frac{7}{8(s+1)}$$

Thus

$$\mathcal{L}^{-1}\{y\} = -\frac{1}{4}e^t + \frac{3}{8}e^{3t} + \frac{7}{8}e^{-t}$$

$y'' - 2y' - 3y = e^t$
$y(0) = 1$, Exact solution is: $y(t) = -\frac{1}{4}e^t + \frac{3}{8}e^{3t} + \frac{7}{8}e^{-t}$
$y'(0) = 0$

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4. (a) (15 pts) Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0 \qquad y(0) = y(\pi) = 0$$

Be sure to consider the cases $\lambda < 0, \lambda = 0,$ and $\lambda > 0.$

Solution:

Case I: $\lambda < 0,$ let $\lambda = -\alpha^2, \alpha \neq 0.$ The DE becomes

$$y'' - \alpha^2 y = 0$$

so

$$y(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$$

The boundary conditions imply

$$C_1 + C_2 = 0$$
$$C_1 e^{\alpha\pi} + C_2 e^{-\alpha\pi} = 0$$

, Solution is: $\{C_1 = 0, C_2 = 0\}.$ Thus $y = 0,$ and there are no negative eigenvalues.

Case II: $\lambda = 0.$ The DE becomes

$$y'' = 0$$

so $y = Ax + B.$ The BCs imply $A = B = 0,$ so 0 is not an eigenvalue.

Case III: $\lambda > 0.$ Let $\lambda = \beta^2$ where $\beta \neq 0.$ The DE becomes

$$y'' + \beta^2 y = 0$$

so

$$y(x) = C_1 \sin\beta x + C_2 \cos\beta x$$

Therefore

$$y(0) = C_2 = 0$$

$$y(\pi) = C_1 \sin\beta\pi = 0$$

For a nontrivial solution we must have $\beta = n, n = 1, 2, \ldots$ or

$$\lambda = n^2, n = 1, 2, \ldots$$

These are the eigenvalues. The eigenfuntions are

$$y_n(x) = D_n \sin nx$$

4(b) (10 pts) Compute the Wronskian for the solutions to the differential equation

$$y'' - 2y' + 2y = 0$$

Solution: The characteristic equation is

$$r^2 - 2r + 2 = 0$$

so

$$r = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm i$$

Hence the solution is: $y(x) = C_1 e^x \sin x + C_2 e^x \cos x$

$$W[e^x \sin x, e^x \cos x] = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \sin x + e^x \cos x & e^x \cos x - e^x \sin x \end{vmatrix} = -e^{2x}$$

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or

$$W[e^x \cos x, e^x \sin x] = -\begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \sin x + e^x \cos x & e^x \cos x - e^x \sin x \end{vmatrix} = e^{2x}$$

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5. (a) (15 pts) Find the first five nonzero terms of the Fourier cosine series for the function

$$f(x) = \begin{cases} 0 & 0 \le x \le \frac{\pi}{2} \\ -4 & \frac{\pi}{2} \le x \le \pi \end{cases}$$

Be sure to give the Fourier series with these terms in it.

Solution: $L = \pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

Where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} 0\cos nx dx + \int_{\frac{\pi}{2}}^{\pi}(-4)\cos nx dx\right]$$

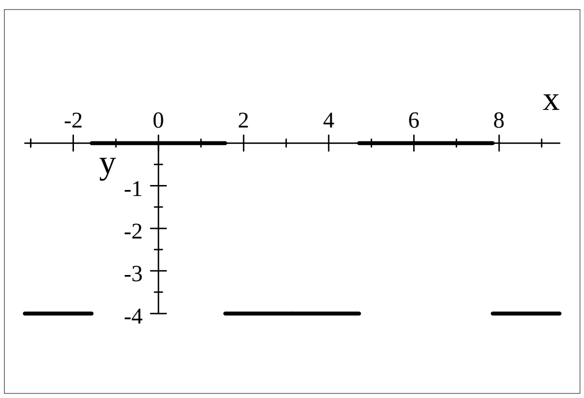
$$= -\frac{8}{n\pi}\sin nx\Big|_{\frac{\pi}{2}}^{\pi} = -\frac{8}{n\pi}\left[\sin n\pi - \sin\left(\frac{n\pi}{2}\right)\right] = \frac{8}{n\pi}\left[\sin\frac{n\pi}{2}\right]$$

$$a_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(-4)dx = -2$$

Thus

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx = -2 + \frac{8}{\pi}\cos x + 0\cos 2x - \frac{8}{3\pi}\cos 3x + 0\cos 4x + \frac{8}{5\pi}\cos 5x + 0\cos 6x - \frac{8}{7\pi}\cos 7x\cdots$$

(b) (10 pts) Sketch the graph of the function represented by the Fourier cosine series in 5 (a) on $-\pi \le x \le 3\pi$.



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6. (25 pts) Solve

$$\begin{aligned} &\text{PDE} \quad u_{xx} = 4u_{tt} \\ &\text{BCS} \quad u_x(0,t) = 0 \qquad u_x(\pi,t) = 0 \\ &\text{ICs} \quad u(x,0) = 0 \qquad u_t(x,0) = -9\cos(4x) + 16\cos(8x) \end{aligned}$$

You must derive the solution. Your solution should not have any arbitrary constants in it.

Solution:

Let $u(x,t) = X(x)T(t)$. Then differentiating and substituting in the PDE yields

$$X''T = 4XT''$$
$$\Rightarrow$$
$$\frac{X''}{X} = 4\frac{T''}{T}$$

Using the argument that the left hand side is purely a function of x and the right hand side is purely a function of t, and the only way that they can be equal is if they are equal to a constant, we get

$$\frac{X''}{X} = 4\frac{T''}{T} = k \qquad k \text{ a constant}$$

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T'' - \frac{1}{4}kT = 0$$

The boundary condition $u_x(0,t) = 0$ implies, since $u_x(x,t) = X'(x)T(t)$ that $X'(0)T(t) = 0.$ We cannot have $T(t) = 0,$ since this would imply that $u(x,t) = 0.$ Thus $X'(0) = 0.$ Similarly, the boundary condition $u_x(\pi,t) = 0$ leads to $X'(\pi) = 0.$

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X'(0) = X'(\pi) = 0$$

For $k > 0,$ the only solution is $X = 0.$ For $k = 0$ we have $X = Ax + B.$ $X'(x) = A,$ so the BCs imply that

$$X(x) = B, \quad B \neq 0$$

is a nontrivial solution corresponding to the eigenvalue $k = 0.$

For $k < 0,$ let $-k = \alpha^2,$ where $\alpha \neq 0.$ Then we have the equation

$$X'' + \alpha^2 X = 0$$

and

$$X(x) = c_1 \sin\alpha x + c_2 \cos\alpha x$$
$$X'(x) = c_1\alpha\cos\alpha x - c_2\alpha\sin\alpha x$$

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$$X'(0) = c_1\alpha = 0$$

so $c_1 = 0.$

$$X'(\pi) = -c_2\alpha\sin\alpha\pi = 0$$

Therefore $\alpha = n,\ n = 1,2,\ldots$ and the solution is

$$k = -n^2 \qquad X_n(x) = a_n\cos nx \qquad n = 1,2,3,\ldots$$

The case $k = 0$ implies that the equation for T becomes $T'' = 0,$ so $T = At + B.$ The initial condition $u(x,0) = 0$ implies $X(x)T(0) = 0$ so that $T(0) = 0.$ Thus $B = 0$ and $T = At$ for $k = 0.$
Substituting the values of $k = -n^2$ into the equation for $T(t)$ leads to

$$T'' + \frac{n^2}{4}T = 0$$

which has the solution

$$T_n(t) = B_n\sin\frac{nt}{2} + C_n\cos\frac{nt}{2}, \quad n = 1,2,3,\ldots$$

The initial condition $u(x,0) = 0$ implies $X(x)T(0) = 0$ so that $T(0) = 0.$ Thus $c_n = 0.$

We now have the solutions

$$u_n(x,t) = A_n\cos nx\sin\frac{nt}{2} \qquad n = 1,2,3,\ldots$$
$$u_0(x,t) = A_0t$$

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x,t) = \sum_{n=0}^{\infty}u_n(x,t) = A_0t + \sum_{n=1}^{\infty}A_n\cos nx\sin\frac{nt}{2}$$

satisfies the PDE, the boundary conditions, and the first initial condition. Since

$$u_t(x,t) = A_0 + \sum_{n=1}^{\infty}A_n\left(\frac{n}{2}\right)\cos nx\cos\frac{nt}{2}$$

the last initial condition leads to

$$u_t(x,0) = -9\cos(4x) + 16\cos(8x) = A_0 + \sum_{n=1}^{\infty}A_n\left(\frac{n}{2}\right)\cos nx.$$

Matching the cosine terms on both sides of this equation leads to

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$A_4\left(\frac{4}{2}\right) = -9$ so that $A_4 = -\frac{9}{2}$ and $A_8\left(\frac{8}{2}\right) = 16$ so that $A_8 = 4$. All of the other constants must be zero, since there are no cosine terms or constant terms on the left to match with. Thus

$$u(x,t) = -\frac{9}{2}\cos 4x \sin 2t + 4\cos 8x \sin 4t$$

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7. (a) (15 pts) Find the power series solution to

$$y'' + 2xy' - 4y = 0$$

near $x = 0$. Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first six nonzero terms of the solution.

Solution:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

Substituting into the DE we have

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + 2\sum_{n=1}^{\infty} a_n (n) x^n - 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n (2n-4) x^n - 4a_0 = 0$$

We now shift the first summation in the last equation above by letting $k = n-2$ or $n = k+2$.

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{n=1}^{\infty} a_n (2n-4) x^n - 4a_0 = 0$$

We write the first term of the first sum separately and replace k and n by m to get

$$\sum_{m=1}^{\infty} [a_{m+2}(m+2)(m+1) + a_m(2m-4)]x^m - 4a_0 + a_2(2)(1) = 0$$

Therefore

$$a_2 = 2a_0$$

and the recurrence relation is

$$a_{m+2}(m+2)(m+1) + a_m(2m-4) = 0 \quad m = 1, 2, \ldots$$

or

$$a_{m+2} = \frac{2(2-m)}{(m+2)(m+1)} a_m \quad m = 1, 2, \ldots$$

$m = 1 \Rightarrow$

$$a_3 = \frac{2(1)}{(3)(2)} a_1 = \frac{1}{3} a_1$$

$m = 2 \Rightarrow$

$$a_4 = 0$$

$m = 3 \Rightarrow$

$$a_5 = \frac{2(-1)}{(5)(4)} a_3 = \frac{-2}{(5)(4)(3)} a_1$$

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$m = 4 \Rightarrow a_6 = 0.$ In fact all of the even coefficients are zero except for a_0 and a_2.
$m = 5 \Rightarrow$

$$a_7 = \frac{2(-3)}{(7)(6)} a_5 = \frac{(2)(2)(3)}{(7)(6)(5)(4)(3)} a_1$$

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \\ &= a_0[1+2x] + a_1\left[x + \frac{1}{3}x^3 + \frac{-2}{(5)(4)(3)}x^5 + \frac{(2)(2)(3)}{(7)(6)(5)(4)(3)}x^7 + \cdots\right] \end{aligned}$$

(b) (10 pts) Solve

$$y'' + 2y' - 3y = 5\sin 3t \quad y(0) = -\frac{1}{2} \quad y'(0) = -\frac{5}{6}$$

Solution: The homogeneous equation is

$$y'' + 2y' - 3y = 0$$

so the characteristic equation is

$$p(r) = r^2 + 2r - 3 = (r+3)(r-1) = 0$$

Thus $r = 1, -3$ and the homogeneous solution if

$$y_h(x) = C_1 e^t + C_2 e^{-3t}$$

We present two methods for finding a particular solution for

$$y'' + 2y' - 3y = 5\sin 3t \qquad (*)$$

I. Complex Variable method: Consider a companion equation

$$v'' + 2v' - 3v = 5\cos 3t \qquad (**)$$

Multiplying the $(*)$ by i, adding the result to $(**)$ and letting $w = v + iy$ we get

$$w'' + 2w' - 3w = 5(\cos 3t + i\sin 3t) = 5e^{3it}$$

Hence

$$\begin{aligned} w_p &= \frac{5e^{3it}}{p(3i)} = \frac{5}{-12+6i}e^{3it} = -\frac{5}{6}\left(\frac{1}{2-i}\right)e^{3it} \\ &= -\frac{5}{6}\left(\frac{1}{2-i}\right)\left(\frac{2+i}{2+i}\right)(\cos 3t + i\sin 3t) = -\frac{1}{6}(2\cos 3t - \sin 3t + i[\cos 3t + 2\sin 3t]) \end{aligned}$$

Thus

$$y_p = -\frac{1}{6}[\cos 3t + 2\sin 3t]$$

II. Alternative Method:

$$y_p(t) = A\cos 3t + B\sin 3t$$

Then

$$\begin{aligned} y_p'(t) &= -3A\sin 3t + 3B\cos 3t \\ y_p''(t) &= -9A\cos 3t - 9B\sin 3t \end{aligned}$$

$$y_p''(t) + 2y_p'(t) - 3y_p(t) = -12A\cos 3t - 12B\sin 3t - 6A\sin 3t + 6B\cos 3t = 5\sin 3t$$

Thus

$$-12A + 6B = 0$$
$$-6A - 12B = 5$$

, Solution is: $\left\{B = -\frac{1}{3}, A = -\frac{1}{6}\right\}$ so

$$y_p = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t$$

as before. Therefore

$$y_g(t) = C_1 e^t + C_2 e^{-3t} - \frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t$$

Applying the initial conditions we have

$$y_g(0) = C_1 + C_2 - \frac{1}{6} = -\frac{1}{2}$$
$$y_g'(0) = C_1 - 3C_2 - 1 = -\frac{5}{6}$$

$$C_1 + C_2 = -\frac{1}{3}$$
$$C_1 - 3C_2 = \frac{1}{6}$$

, Solution is: $\left\{C_2 = -\frac{1}{8}, C_1 = -\frac{5}{24}\right\}$

Therefore

$$y(t) = -\frac{5}{24}e^t + -\frac{1}{8}e^{-3t} - \frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t$$

$y'' + 2y' - 3y = 5\sin 3t$

$y(0) = -\frac{1}{2}$, Exact solution is: $y(t) = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t - \frac{5}{24}e^t - \frac{1}{8}e^{-3t}$,

$y'(0) = -\frac{5}{6}$

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Part II: **Choose only one question**.

8. (a) (13 pts) Given that x and e^x are solutions of the homogeneous equation associated with

$$(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}$$

use this fact to solve the nonhomogeneous equation.

Solution: We use Variation of Parameters to solve the equation. Therefore

$$y(x) = xv_1 + v_2 e^x$$

The two equations for v_1' and v_2' are

$$\begin{aligned} xv_1' + v_2' e^x &= 0 \\ v_1' + v_2' e^x &= \frac{f(x)}{a(x)} = -\frac{2(x-1)^2 e^{-x}}{(x-1)} = 2(1-x)e^{-x} \end{aligned}$$

Then

$$v_1' = \frac{\begin{vmatrix} 0 & e^x \\ 2(1-x)e^{-x} & e^x \end{vmatrix}}{\begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}} = \frac{-2(1-x)}{(x-1)e^x} = +2e^{-x}$$

$$v_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & 2(1-x)e^{-x} \end{vmatrix}}{\begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}} = \frac{2x(1-x)e^{-x}}{(x-1)e^x} = -2xe^{-2x}$$

Thus

$$v_1 = -2e^{-x} + C_1$$

$$v_2 = \int \left(-2xe^{-2x}\right) dx = xe^{-2x} + \frac{1}{2}e^{-2x} + C_2$$

Thus

$$\begin{aligned} y(x) &= C_1 x + C_2 e^x - 2xe^{-x} + xe^{-x} + \frac{1}{2}e^{-x} \\ &= C_1 x + C_2 e^x - xe^{-x} + \frac{1}{2}e^{-x} \end{aligned}$$

SNB check $(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}$, Exact solution is: $y(x) = -\frac{1}{2}e^{-x}(2x-1) + C_1 x + C_2 e^x$

8 (b) (i) (8 pts) Find the Fourier sine series for $f(x) = x$ on $0 < x < \pi$

Solution: $L = \pi$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} x \sin nx dx = \frac{2}{n\pi}\left[-x\cos nx|_0^{\pi} + \int_0^{\pi} \cos nx dx\right] \\ &= \frac{2}{n\pi}\left[(-1)^{n+1}\pi + \frac{\sin nx}{n}\Big|_0^{\pi}\right] = \frac{2(-1)^{n+1}}{n}, \quad n = 1, 2, \ldots \end{aligned}$$

Thus

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$$f(x) = x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

8 (b) (ii) (4 pts) Given that

$$\int_0^{\pi} x^2 dx = \frac{\pi}{2} \sum_{n=1}^{\infty} A_n^2$$

where the $A_n s$ are the Fourier coefficients in the expansion in 8 (b) (i) show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Solution: From the given formula with the value of A_n calculated above

$$\int_0^{\pi} x^2 dx = \frac{\pi}{2} \sum_{n=1}^{\infty} 4\left[\frac{(-1)^{n+1}}{n} \right]^2 = 2\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

But $\int_0^{\pi} x^2 dx = \frac{1}{3}\pi^3$ so we have

$$\frac{1}{3}\pi^3 = 2\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Which is the result.

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9. (a) (15 pts) Find the eigenvalues and eigenfunctions for

$$x^2y'' + xy' + \lambda y = 0 \qquad y(1) = y(e^{\pi}) = 0$$

Be sure to consider the cases $\lambda < 0, \lambda = 0,$ and $\lambda > 0.$

Solution:

Note that this is an Euler equation!! Since $p = 1, q = \lambda$ the indicial equation is

$$r^2 + (p-1)r + q = r^2 - \lambda = 0$$

I. $\lambda < 0.$ Let $\lambda = -\alpha^2$ where $\alpha \neq 0.$ Then the indicial equation is

$$r^2 - \alpha^2 = 0$$

so $m = \pm\alpha.$ Thus

$$y = c_1x^{\alpha} + c_2x^{-\alpha}$$

The boundary conditions imply

$$c_1 + c_2 = 0$$
$$c_1e^{\alpha\pi} + c_2e^{-\alpha\pi} = 0$$

This leads to $c_1 = c_2 = 0,$ so $y = 0$ and there are no negative eigenvalues.

II. $\lambda = 0.$ The DE for this case is

$$x^2y'' + xy' = 0$$

or

$$xy'' + y' = 0$$

Let $v = y'$ so we have

$$xv' + v = 0$$
$$(xv)' = 0$$

Then $v = \frac{c_1}{x},$ and $y = c_1 \ln x + c_2.$ The BCs lead to

$$c_1 + c_2 = 0$$
$$c_1 \ln e^{\pi} + c_2 = 0$$

or

$$c_1 + c_2 = 0$$
$$\pi c_1 + c_2 = 0$$

Again we have $c_1 = c_2 = 0,$ so $\lambda = 0$ is not an eigenvalue.

III. $\lambda > 0.$ Let $\lambda = \beta^2$ where $\beta \neq 0.$ The indicial equation is

$$r^2 + \beta^2 = 0$$

so

$$r = \pm\beta i$$

Thus

$$y = c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)$$

The BCs imply

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$$c_1 = 0$$

$$c_2 \text{ is arbitrary}$$

$$\sin(\beta\pi) = 0 \Rightarrow \beta = n \text{ for } n = 1,2,\ldots$$

Eigenvalues:

$$\lambda_n = \beta^2 = n^2 \; n = 1,2,\ldots$$

Eigenfunctions:

$$y_n = c_n \sin(n \ln x)$$

(b) (10 pts) Show that if the equation

$$\left(\frac{y^2}{2} + 2ye^x\right)dx + (y + e^x)dy = 0$$

is multiplied by e^x, then the resulting equation is exact and then solve this equation.
Solution: Multiplying by e^x we have

$$\left(\frac{y^2e^x}{2} + 2ye^{2x}\right)dx + \left(ye^x + e^{2x}\right)dy = 0$$

so

$$M_y = \frac{\partial\left(\frac{y^2e^x}{2} + 2ye^{2x}\right)}{\partial y} = ye^x + 2e^x = N_x = \frac{\partial\left(ye^x + e^{2x}\right)}{\partial x}$$

so the multiplied equation is now exact. Hence there exists $f(x,y)$ such that

$$f_x = M = \frac{y^2e^x}{2} + 2ye^{2x}$$

$$f_y = N = ye^x + e^{2x}$$

Starting with f_y and integrating with respect to y we have

$$f(x,y) = \frac{y^2}{2}e^x + ye^{2x} + g(x)$$

so

$$f_x = \frac{y^2}{2}e^x + 2ye^{2x} + g'(x) = M = \frac{y^2e^x}{2} + 2ye^{2x}$$

Thus $g'(x) = 0$ so $g(x) = k$ where k is a constant. Thus the solution is given by

$$\frac{y^2}{2}e^x + ye^{2x} = C$$

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Table of Laplace Transforms

$f(t)$	$\hat{f}(s)$	
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1 \quad s > 0$
$\sin ax$	$\frac{a}{s^2 + a^2}$	$s > a$
$\cos ax$	$\frac{s}{s^2 + a^2}$	$s > a$
$e^{-bt}f(t)$	$\hat{f}(s+b)$	
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} \hat{f}(s)$	

Table of Integrals

$$\int \tan x \sin x dx = -\sin x + \ln(\sec x + \tan x) + C$$

$$\int x e^x dx = x e^x - e^x + C$$

$$\int x \sin x dx = \sin x - x \cos x + C$$

$$\int x \cos x dx = \cos x + x \sin x + C$$

$$\int \sin ax \sin bx dx = \frac{1}{2(-a+b)} \sin(-a+b)x - \frac{1}{2(a+b)} \sin(a+b)x + C \quad a^2 \neq b^2$$

$$\int \sin ax \cos bx dx = -\frac{1}{2} \frac{\cos(a+b)x}{a+b} + \frac{1}{2} \frac{\cos(-a+b)x}{-a+b} + C \quad a^2 \neq b^2$$

$$\int \cos ax \cos bx dx = \frac{1}{2(-a+b)} \sin(-a+b)x + \frac{1}{2(a+b)} \sin(a+b)x + C \quad a^2 \neq b^2$$

$$\int \ln x dx = x \ln x - x + C$$