Name__________________________ Instructor _______________________

Ma 221 **Final Exam Solutions** **12/22/03**

Print Name: _________________________________ **ID**: __________________________

Lecture Section: _________

I pledge my honor that I have abided by the Stevens Honor System.

____

This exam consists of 7 problems. The point value of each problem is indicated. The total number of points is 175, which will be scaled to 200 points after grading.

If you need more work space, continue the problem you are doing on the **other side of the page it is on**. Be sure that you do all problems.

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

Total Score ________

Name___________________________ Instructor _______________________

1. Solve the initial value problem

(a) (8 pts)

$$y' - 4x = -2xy, \;\; y(0) = 4$$

Solution: We write the equation as the first order linear DE

$$y' + 2xy = 4x$$

Then an integrating factor is

$$e^{\int 2x dx} = e^{x^2}$$

Multipying the DE by this yields

$$y' e^{x^2} + 2xe^{x^2} = \frac{d}{dx}\left(e^{x^2} y\right) = 4xe^{x^2}$$

Integrating we have

$$e^{x^2} y = 2e^{x^2} + C$$

or

$$y(x) = 2 + Ce^{-x^2}$$

The initial condition implies

$$y(0) = 2 + C = 4$$

so $C = 2.$ Thus

$$y(x) = 2 + 2e^{-x^2}$$

SNB check: $\begin{matrix} y' + 2xy = 4x \\ y(0) = 4 \end{matrix}$, Exact solution is: $y(x) = 2 + 2e^{-x^2}$

(b) (7 pts) Solve

$$y' = \frac{x^2}{y}$$

Solution: This equation is separable and can be written as

$$ydy = x^2 dx$$

so

$$\frac{y^2}{2} = \frac{1}{3}x^3 + C$$

1 (c) (10 pts) Solve the initial value problem

$$x^2 y'' + xy' - 4y = 0, \;\; y(1) = 3, \;\; y'(1) = 10 \;\; x > 0$$

Solution: This is an Euler equation with $p = 1$ and $q = -4.$ The indicial equation is

$$r^2 + (p-1)r + q = r^2 - 4 = 0$$

so $r = \pm 2$ and

$$y(x) = c_1 x^2 + c_2 x^{-2}$$

and

$$y'(x) = 2c_1 x - 2c_2 x^{-3}$$

The inital conditions imply

Name__________________________ Instructor _______________________

$$c_1 + c_2 = 3$$
$$2c_1 - 2c_2 = 10$$

so that $c_1 = 4, c_2 = -1$ and

$$y(x) = 4x^2 - x^{-2}$$

2. (a) (15 pts) Find a general solution of

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$$

Solution: The characteristic equation is

$$p(r) = r^2 - 3r + 2 = (r-2)(r-1) = 0$$

Thus $r = 1, 2$ and $y_h = c_1e^{2x} + c_2e^x$. We must find a particular solution for each term on the right hand side.

For $3e^{-x}$ we have

$$y_{p_1} = \frac{ke^{\alpha x}}{p(\alpha)} = \frac{3e^{-x}}{p(-1)} = \frac{3e^{-x}}{6} = \frac{1}{2}e^{-x}$$

For $-10\cos 3x$ we consider the equations

$$y'' - 3y' + 2y = -10\cos 3x$$
$$v'' - 3v' + 2v = -10\sin 3x$$

Multiplying the second equation by i, adding the two equations together and letting $w = y + iv$ we get

$$w'' - 3w' + 2w = -10(\cos 3x + i\sin 3x) = -10e^{3ix}$$

Thus

$$w_p = \frac{-10e^{3ix}}{p(3i)} = -\frac{10e^{3ix}}{-7-9i}$$

The particular solution we are looking for is the real part of w_p. Thus

$$w_p = \frac{10e^{3ix}}{7+9i} \times \left(\frac{7-9i}{7-9i}\right) = \frac{10(7-9i)(\cos 3x + i\sin 3x)}{49+81} = \frac{(7-9i)(\cos 3x + i\sin 3x)}{13}$$

so

$$y_{p_2} = \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x$$

Hence

$$y(x) = y_h + y_{p_1} + y_{p_2} = c_1e^{2x} + c_2e^x + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x$$

SNB Check: $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$, Exact solution is:
$C_{15}e^x + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x + \frac{1}{2e^x} + C_{16}e^{2x}$

2(b) (10 pts) Find a general solution of

$$y'' + 2y' + y = \frac{e^{-x}}{x}$$

Solution: We use Variation of parameters. The characteristic equation is $p(r) = r^2 + 2r + 1 = (r+1)^2$. Thus $y_h = c_1e^{-x} + c_2xe^{-x}$. We let

Name___________________________ Instructor ________________________

$$y_p = v_1(x)e^{-x} + v_2(x)xe^{-x}$$

The equations for v_1' and v_2' are

$$v_1'e^{-x} + v_2'(xe^{-x}) = 0$$
$$-v_1'e^{-x} + v_2'(e^{-x} - xe^{-x}) = \frac{e^{-x}}{x}$$

Then

$$v_1' = \frac{\begin{vmatrix} 0 & xe^{-x} \\ \frac{e^{-x}}{x} & e^{-x} - xe^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}} = -\frac{e^{-2x}}{e^{-2x}} = -1$$

$$v_2' = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{e^{-x}}{x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}} = \frac{\frac{e^{-2x}}{x}}{e^{-2x}} = \frac{1}{x}$$

Hence

$$v_1 = -x$$
$$v_2 = \ln x$$

$$y_p = v_1(x)e^{-x} + v_2(x)xe^{-x} = -xe^{-x} + xe^{-x}\ln x$$

Since xe^{-x} is a homogeneous solution, we may ignore the $-xe^{-x}$ in y_p and

$$y = y_h + y_p = c_1e^{-x} + c_2xe^{-x} + xe^{-x}\ln x$$

SNB check: $y'' + 2y' + y = \frac{e^{-x}}{x}$, Exact solution is: $C_{30}e^{-x} - \frac{x}{e^x} + C_{29}xe^{-x} + x\frac{\ln x}{e^x}$

3. (a) (10 pts) Find

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+8s+20}\right\}$$

Solution:

$$\frac{s-2}{s^2+8s+20} = \frac{s-2}{(s+4)^2+4} = \frac{s+4}{(s+4)^2+4} + \frac{-6}{(s+4)^2+4}$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+8s+20}\right\} = e^{-4t}\cos 2t - 3e^{-4t}\sin 2t$$

.

(b) (15 pts) Use Laplace Transforms to solve:

$$y' - y = -2\cos t, \quad y(0) = 1$$

Solution: Taking the Laplace transform of both sides we have

Name__________________________ Instructor ______________________

$$s\mathcal{L}\{y\} - y(0) - \mathcal{L}\{y\} = -\frac{2s}{s^2+1}$$

or

$$(s-1)\mathcal{L}\{y\} = -\frac{2s}{s^2+1} + 1$$

so that

$$\mathcal{L}\{y\} = -\frac{2s}{\left(s^2+1\right)(s-1)} + \frac{1}{s-1}$$

To invert $\frac{-2s}{(s^2+1)(s-1)}$ we use partial fractions.

We may separate $\frac{-2s}{(s^2+1)(s-1)}$ by writing it as

$$\frac{-2s}{\left(s^2+1\right)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1}$$

$\frac{-2s}{(s^2+1)(s-1)} = \frac{s-1}{s^2+1} - \frac{1}{s-1}$

Again $C = -1.$ Letting $s = 0$ we have

$$0 = B + 1 \Rightarrow B = -1$$

Letting $s = -1$ we have

$$\frac{2}{2(-2)} = \frac{A-1}{2} - \frac{-1}{-2}$$

or

$$-\frac{1}{2} = \frac{A}{2} - \frac{1}{2} - \frac{1}{2}$$

so $A = 1.$ Thus

$$\frac{-2s}{\left(s^2+1\right)(s-1)} = \frac{s-1}{s^2+1} + \frac{-1}{s-1}$$

and we have

$$\mathcal{L}\{y\} = -\frac{2s}{\left(s^2+1\right)(s-1)} + \frac{1}{s-1} = \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

and

$$y(x) = \cos t - \sin t$$

Another Approach:

$$\frac{-2s}{\left(s^2+1\right)(s-1)} = \frac{-2s}{(s+i)(s-i)(s-1)} = \frac{A}{s+i} + \frac{B}{s-i} + \frac{C}{s-1}$$

Then $C = \frac{-2}{1+1} = -1,\ A = \frac{2i}{(-i-i)(-i-1)} = \frac{1}{(i+1)}, B = \frac{-2i}{(i+i)(i-1)} = \frac{-1}{(i-1)}$

Now

$$A = \frac{1}{(i+1)} \times \frac{(i-1)}{(i-1)} = \frac{-(i-1)}{2}$$

$$B = \frac{-1}{(i-1)} \times \frac{(i+1)}{(i+1)} = \frac{(i+1)}{2}$$

So

Name__________________________ Instructor _______________________

$$\frac{-2}{\left(s^2+1\right)(s-1)} = -\left(\frac{i-1}{2}\right)\left(\frac{1}{s+i}\right) + \left(\frac{i+1}{2}\right)\left(\frac{1}{s-i}\right) + \frac{-1}{s-1}$$

Hence

$$\begin{aligned}
\mathcal{L}\{y\} &= -\frac{2}{\left(s^2+1\right)(s-1)} + \frac{1}{s-1} = -\left(\frac{i-1}{2}\right)\left(\frac{1}{s+i}\right) + \left(\frac{i+1}{2}\right)\left(\frac{1}{s-i}\right) + \frac{-1}{s-1} + \frac{1}{s-1} \\
&= -\left(\frac{i-1}{2}\right)\left(\frac{1}{s+i}\right) + \left(\frac{i+1}{2}\right)\left(\frac{1}{s-i}\right)
\end{aligned}$$

$$\begin{aligned}
y(x) &= \mathcal{L}^{-1}\left\{\frac{-2}{\left(s^2+1\right)(s-1)} + \frac{1}{s-1}\right\} = \frac{-(i-1)}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+i}\right\} + \frac{(i+1)}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-i}\right\} \\
&= \frac{1-i}{2}e^{-it} + \frac{1-i}{2}e^{it} \\
&= \left(\frac{1-i}{2}\right)(\cos t - i\sin t) + \left(\frac{1-i}{2}\right)(\cos t + i\sin t) \\
&= \cos t - \sin t
\end{aligned}$$

SNB check: $\begin{aligned} y' - y &= -2\cos t \\ y(0) &= 1 \end{aligned}$, Exact solution is: $\cos t - \sin t$

4. (a) (15 pts) Find the eigenvalues and eigenfunctions for

$$y'' + 2y' + (1+\lambda)y = 0 \qquad y(0) = y(1) = 0$$

Be sure to consider all possible values of λ.
Solution:
The characteristic equation is $r^2 + 2r + 1 + \lambda = 0$ and has solutions

$$r = -\frac{-2 \pm \sqrt{4 - 4(1+\lambda)}}{2} = -1 \pm \sqrt{-\lambda}$$

Thus there are 3 cases to consider: $\lambda > 0, \lambda = 0,$ and $\lambda < 0.$ We deal with each separately.
I. $\lambda < 0.$ Let $-\alpha^2 = \lambda,$ where $\alpha \neq 0,$ so that $r = -1 \pm \alpha$ and

$$y(x) = c_1 e^{(-1+\alpha)x} + c_2 e^{(-1-\alpha)x}$$

The boundary conditions imply

$$\begin{aligned}
y(0) &= c_1 + c_2 = 0 \\
y(1) &= c_1 e^{(-1+\alpha)} + c_2 e^{-(1+\alpha)}
\end{aligned}$$

so that $c_1 = c_2 = 0.$ Therefore $y = 0$ and there are no eigenvalues for $\lambda > 0.$

II. $\lambda = 0.$ There is only one repeated root $r = -1$ and $y = c_1 e^{-x} + c_2 x e^{-x}.$ Then $y(0) = c_1 = 0$ and $y(1) = c_2 e^{-1} = 0,$ so $c_2 = 0,$ and $y = 0.$ Again there are not eigenvalues for this case.

III. $\lambda > 0.$ Let $\beta^2 = \lambda > 0.$ Then $r = -1 \pm i\beta$ and

$$y(x) = c_1 e^{-x}\sin\beta x + c_2 e^{-x}\cos\beta x$$

The boundary conditions then yield

$$y(0) = c_2 = 0$$
$$y(1) = c_1 e^{-1} \sin\beta = 0$$

$\Rightarrow$

$$\sin\beta = 0 \quad \text{or} \quad \beta = n\pi, \;\; n = 1,2,\ldots$$

Then

$$\lambda = -\beta^2 = -n^2\pi^2, \quad n = 1,2,\ldots$$

are the eigenvalues, and the eigenfunctions are

$$y_n(x) = a_n e^{-x} \sin n\pi x$$

4(b) (10 pts) Use separation of variables, $u(x,t) = X(x)T(t)$, to find two ordinary differential equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$-3x^4t^3 \frac{\partial^2 u}{\partial x^2} + (x+6)^5(t-2)^3 \frac{\partial u}{\partial t} = 0.$$

Note: Do **not** solve these ordinary differential equations.
Solution:

$$\frac{\partial u}{\partial x} = X'(x)T(t)$$
$$\frac{\partial u}{\partial x} = X''(x)T(t)$$
$$\frac{\partial u}{\partial t} = X(x)T'(t)$$

So the DE implies

$$-3x^4t^3X''T + (x+6)^5(t-2)^3XT' = 0$$

or

$$\frac{3x^4X''}{(x+6)^5X} = \frac{(t-2)^3T'}{t^3T} = k$$

where k is a constant. The two DEs are

$$3x^4X'' - k(x+6)^5X = 0$$
$$(t-2)^3T' - kt^3T = 0$$

5. (a) (15 pts) Find the first five nonzero terms of the Fourier *sine* series for the function

$$f(x) = \begin{cases} -2 & 0 \le x \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < x \le \frac{\pi}{2} \end{cases}$$

Be sure to give the Fourier series with these terms in it.
Solution: $L = \frac{\pi}{2}$.

$$f(x) = \sum_{1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{1}^{\infty} a_n \sin(2nx)$$

and

Name___________________________ Instructor ________________________

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)\sin(2nx)dx \quad n = 1,2,3,\ldots$$

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{4}}(-2)\sin(2nx)dx = +\frac{4}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \cos 0\right] = \frac{4}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] \quad n = 1,2,3,\ldots$$

Therefore

$$a_1 = \frac{4}{\pi}\left[\cos\left(\frac{\pi}{2}\right) - 1\right] = -\frac{4}{\pi}$$
$$a_2 = \frac{4}{2\pi}[\cos\pi - 1] = \frac{-8}{2n} = -\frac{4}{\pi}$$
$$a_3 = \frac{4}{3\pi}\left[\cos\left(\frac{3\pi}{2}\right) - 1\right] = -\frac{4}{3\pi}$$
$$a_4 = \frac{4}{4\pi}[\cos(2\pi) - 1] = 0$$
$$a_5 = \frac{4}{5\pi}\left[\cos\left(\frac{5\pi}{2}\right) - 1\right] = -\frac{4}{5\pi}$$
$$a_6 = \frac{4}{6\pi}[\cos(3\pi) - 1] = -\frac{4}{3\pi}$$

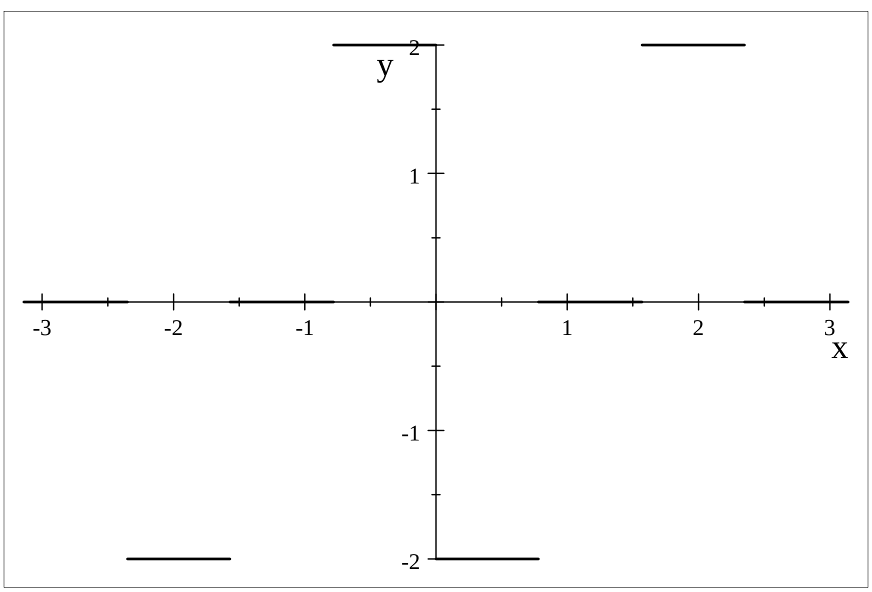
Thus

$$f(x) = \sum_1^\infty a_n\sin(2nx) = a_1\sin 2x + a_2\sin 4x + \cdots$$
$$= \left(-\frac{4}{\pi}\right)\sin 2x + \left(-\frac{4}{\pi}\right)\sin 4x + \left(-\frac{4}{3\pi}\right)\sin 6x + 0\sin 8x + \left(-\frac{4}{5\pi}\right)\sin 10x + \left(-\frac{4}{3\pi}\right)\sin 12x + \cdots$$

(b) (10 pts) Sketch the graph of the function represented by the Fourier cosine series in 5 (a) on $-\pi \le x \le \pi$.

$\frac{\pi}{4} = 0.78540$, $\frac{\pi}{2} = 1.5708$ $\frac{3\pi}{4} = 2.3562$

$(0,-2,.78,-2)$



Name__________________________ Instructor _______________________

6 (25 pts) Solve

$$\text{PDE} \quad u_{xx} = 9u_t$$
$$\text{BCs} \quad u(0,t) = 0 \qquad u_x(\pi,t) = 0$$
$$\text{ICs} \quad u(x,0) = 12\sin\left(\frac{x}{2}\right) - 3\sin\left(\frac{9x}{2}\right)$$

You must derive the solution. Your solution should not have any arbitrary constants in it.
Solution: Let $u(x,t) = X(x)T(t)$. Then the PDE implies

$$X''T = 9XT'$$

or

$$\frac{X''}{X} = 9\frac{T'}{T} = k, \text{ a constant}$$

For a nontrivial solution we set $k = -\lambda^2$ where $\lambda \neq 0$ and get

$$X'' + \lambda^2 X = 0, \quad X(0) = X'(\pi) = 0$$
$$T' + \frac{1}{9}\lambda^2 T = 0$$

The solution to the X equation is

$$X(x) = c_1 \sin\lambda x + c_2 \cos\lambda x$$

The condtion $x(0) = 0 \Rightarrow c_2 = 0$. Also

$$X'(x) = c_1\lambda\cos\lambda x$$

so

$$X'(\pi) = c_1\lambda\cos\lambda\pi = 0$$

Therefore

$$\lambda\pi = \frac{(2n+1)\pi}{2}, \quad n = 0,1,2,\ldots$$

or

$$\lambda = \frac{2n+1}{2}, \quad n = 0,1,2,\ldots$$

and

$$X_n(x) = a_n \sin\left(\frac{2n+1}{2}\right)x \quad n = 0,1,2,\ldots$$

For $t(t)$ we have

$$T' + \left(\frac{1}{9}\right)\left(\frac{2n+1}{4}\right)^2 T = 0$$

so

$$T_n(t) = b_n e^{-\left(\frac{1}{9}\right)\left(\frac{2n+1}{4}\right)^2 t}$$

and

$$u_n(x,t) = D_n \sin\left(\frac{2n+1}{2}\right)x e^{-\left(\frac{1}{9}\right)\left(\frac{2n+1}{4}\right)^2 t} \quad n = 0,1,2,\ldots$$

We let

Name__________________________ Instructor ______________________

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} D_n \sin\left(\frac{2n+1}{2}\right) x e^{-\left(\frac{1}{9}\right)\left(\frac{2n+1}{4}\right)^2 t}$$

Then

$$u(x,0) = \sum_{n=0}^{\infty} D_n \sin\left(\frac{2n+1}{2}\right)x = 12\sin\left(\frac{x}{2}\right) - 3\sin\left(\frac{9}{2}\right)x$$

Hence $D_0 = 12, D_4 = -3$ and $D_n = 0\ n \neq 0, 4$. The final solution is therefore

$$u(x,t) = 12\sin\left(\frac{x}{2}\right)e^{-\left(\frac{1}{9}\right)\frac{1}{16}t} - 3\sin\left(\frac{9x}{2}\right)e^{-\left(\frac{1}{9}\right)\frac{81}{16}t}$$

7. (a) (15 pts) Find the power series solution to

$$(x^2+1)y'' + xy' - y = 0$$

near $x = 0$. Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first six nonzero terms of the solution.

Solution:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

The DE implies

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^n + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n(n)x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Combining the there sums that have x^n in them, and shifting the sum with x^{n-2} in it by letting $k = n-2$ or $n = k+2$ we have

$$-a_0 - a_1 x + a_1 x + \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{n=2}^{\infty} a_n[(n)(n-1) + n - 1]x^n = 0$$

Or after replacing k and n by the "dummy" place keeper m

$$-a_0 + a_2(2)(1) + a_3(3)(2)x + \sum_{m=2}^{\infty}\left\{a_{m+2}(m+2)(m+1) + a_m\left(m^2-1\right)\right\}x^m = 0$$

Thus

$$a_2 = \frac{1}{2}a_0$$

$$a_3 = 0$$

$$a_{m+2} = -\frac{m^2-1}{(m+2)(m+1)}a_m = -\frac{m-1}{m+2}a_m \quad m = 2, 3, \ldots$$

Name___________________________ Instructor _______________________

$$m = 2 \Rightarrow a_4 = -\frac{1}{4}a_2 = -\frac{1}{8}a_0$$
$$m = 3 \Rightarrow a_5 = 0$$
$$m = 4 \Rightarrow a_6 = -\frac{3}{6}a_4 = +\frac{1}{16}a_0$$
$$m = 5 \Rightarrow a_7 = 0$$
$$m = 6 \Rightarrow a_8 = -\frac{5}{8}a_6 = -\frac{5}{8(16)}a_0$$

All of the odd coefficients $a_{2j+1} = 0$ for $j \geq 1$. Therefore

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= a_0\left[1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8 + \cdots\right] + a_1 x$$

SNB Check: $(x^2+1)y'' + xy' - y = 0$, Series solution is:
$y(x) = y(0) + y'(0)x + \left(\frac{1}{2}y(0)\right)x^2 + \left(-\frac{1}{8}y(0)\right)x^4 + \left(\frac{1}{16}y(0)\right)x^6 + \left(-\frac{5}{128}y(0)\right)x^8 + O\left(x^9\right)$

Name__________________________ Instructor ______________________

(b) (10 pts) Solve

$$y'' + 2y' + y = t^2 + 1 - e^t \qquad y(0) = 0 \;\; y'(0) = 2$$

Solution: The charactieristic equation is $p(r) = r^2 + 2r + 1 = (r+1)^2 = 0$ so

$$y_h = c_1 e^{-t} + c_2 t e^{-t}$$

Since $p(1) = 4 \neq 0$ a particular solution for $-e^t$ is

$$y_{p_1} = \frac{-e^t}{4}$$

To find a particular solution for the polynomial $t^2 + 1,$ we let

$$y_{p_2} = At^2 + Bt + C$$
$$y'_{p_2} = 2At + B$$
$$y''_{p_2} = 2A$$

and

$$2A + 4At + 2B + At^2 + Bt + C = t^2 + 1$$

Hence $A = 1,\;\; 4A + B = 0$ or $B = -4,$ and $2A + 2B + C = 1$ so $C = 7$ and

$$y_{p_2} = t^2 - 4t + 7$$

Therefore

$$y(t) = y_h + y_{p_1} + y_{p_2} = c_1 e^{-t} + c_2 t e^{-t} - \frac{e^t}{4} + t^2 - 4t + 7$$
$$y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} - \frac{e^t}{4} + 2t - 4$$

The initial conditions imply

$$y(0) = c_1 - \frac{1}{4} + 7 = 0 \Rightarrow c_1 = -\frac{27}{4}$$
$$y'(0) = -c_1 + c_2 - \frac{1}{4} - 4 = 2 \Rightarrow \frac{27}{4} + c_2 - \frac{1}{4} - \frac{16}{4} = 2 \Rightarrow c_2 = 2 - \frac{10}{4} = -\frac{1}{2}$$

Solution is: $\left\{c_2 = -\frac{3}{4}\right\}$

so

$$y(t) = -\frac{27}{4} e^{-t} - \frac{1}{2} t e^{-t} - \frac{e^t}{4} + t^2 - 4t + 7$$

$$y'' + 2y' + y = t^2 + 1 - e^t$$

SNB check" $\quad y(0) = 0 \quad ,$

$$y'(0) = 2$$

Exact solution is: $y(t) = \frac{1}{4} e^t \left(4t^2 e^{-t} - 16 e^{-t} t + 28 e^{-t} - 1\right) - \frac{27}{4} e^{-t} - \frac{1}{2} e^{-t} t$

Name___________________________ Instructor ________________________

Table of Laplace Transforms

$f(t)$	$\hat{f}(s)$	
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1 \quad s > 0$
$\sin ax$	$\frac{a}{s^2 + a^2}$	$s > a$
$\cos ax$	$\frac{s}{s^2 + a^2}$	$s > a$
$e^{-bt} f(t)$	$\hat{f}(s+b)$	
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} \hat{f}(s)$	