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Lecturer _____

Ma 221

Final Exam Solutions

12/20/05

Print Name: _____

Lecture Section: _____

I pledge my honor that I have abided by the Stevens Honor System.

This exam consists of 8 problems. You are to solve all of these problems. The point value of each problem is indicated. The total number of points is 200.

If you need more work space, continue the problem you are doing on the **other side of the page it is on**. Be sure that you do all problems.

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

There are tables giving Laplace transforms and integrals at the end of the exam.

Score on Problem #1 _____

#2 _____

#3 _____

#4 _____

#5 _____

#6 _____

#7 _____

#8 _____

Total Score _____

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1. Solve the initial value problem

(a) (8 pts)

$$\frac{dy}{dx} = \frac{3x \sec y}{\sqrt{1-x^2}}, \quad y(0) = \frac{\pi}{2}$$

Solution: We can separate the variables to obtain:

$$\frac{dy}{\sec y} = \frac{3xdx}{\sqrt{1-x^2}}$$

$$\int \frac{dy}{\sec y} = \int \frac{3xdx}{\sqrt{1-x^2}}$$

In the right side, let $u = 1 - x^2$ so $du = -2xdx$. We then obtain:

$$\int \cos y dy = \int \frac{-3du}{2\sqrt{u}}$$

$$\Rightarrow \sin y = -3\sqrt{1-x^2} + C$$

Applying the initial condition:

$$1 = -3 + C$$

$$\Rightarrow C = 4.$$

So,

$$\sin y = -3\sqrt{1-x^2} + 4$$

$$\Rightarrow y = \arcsin(-3\sqrt{1-x^2} + 4)$$

(b) (7 pts) Solve

$$(3x^2 \sin y - 5x^3)dx + (x^3 \cos y + 5e^y)dy = 0$$

Solution: With $M(x,y) = 3x^2 \sin y - 5x^3$ and $N(x,y) = x^3 \cos y + 5e^y$,

$$\frac{\partial M}{\partial y} = 3x^2 \cos y = \frac{\partial N}{\partial x}$$

So the above DE is exact.

$$\frac{\partial f}{\partial x} = M = 3x^2 \sin y - 5x^3$$

$$\Rightarrow f(x,y) = x^3 \sin y - \frac{5}{4}x^4 + g(y)$$

$$\frac{\partial f}{\partial y} = x^3 \cos y + g'(y) = N = x^3 \cos y + 5e^y$$

$$\Rightarrow g'(y) = 5e^y$$

$$\Rightarrow g(y) = 5e^y + C$$

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$$f(x,y) = x^3 \sin y - \frac{5}{4}x^4 + 5e^y + C$$

So the solution is:

$$x^3 \sin y - \frac{5}{4}x^4 + 5e^y = K$$

1 (c) (10 pts) Find a general solution of

$$y'' + 10y' + 25y = 14e^{-5t}$$

Solution: First we solve the corresponding homogeneous DE:

$$y'' + 10y' + 25y = 0$$

$$p(r) = r^2 + 10r + 25 = 0$$

$$\Rightarrow r = -5 \text{ is a repeated root.}$$

So,

$$y_h = c_1 e^{-5t} + c_2 t e^{-5t}$$

With $\alpha = -5$, $p(-5) = 0$. Also $p'(r) = 2r + 10 \Rightarrow p'(-5) = 0$.

So,

$$y_p = \frac{14t^2 e^{-5t}}{p''(-5)} = \frac{14t^2 e^{-5t}}{2} = 7t^2 e^{-5t}$$

The general solution, therefore, is:

$$y_g = c_1 e^{-5t} + c_2 t e^{-5t} + 7t^2 e^{-5t}$$

2. (a) (12 pts) Find a general solution of

$$y'' - 4y' = t - 3 \sin 2t$$

First we solve the corresponding homogeneous DE:

$$y'' - 4y' = 0$$

$$p(r) = r^2 - 4r = 0$$

$$\Rightarrow r = 0, 4$$

$$y_h = c_1 + c_2 e^{4t}$$

Consider

$$y'' - 4y' = t$$

$$y_{p1} = t(At + B) = At^2 + Bt$$

$$y'_{p1} = 2At + B$$

$$y''_{p1} = 2A$$

Substituting into the DE gives:

$$2A - 8At - 4B = t$$

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$$-8A = 1 \Rightarrow A = -\frac{1}{8}$$

$$2A - 4B = 0 \Rightarrow B = -\frac{1}{16}$$

$$\Rightarrow y_{p1} = -\frac{1}{8}t^2 - \frac{1}{16}t$$

Consider:

$$y'' - 4y' = -3 \sin 2t$$

Approach 1: (with complex variables)

$$w'' - 4w' = -3e^{2it}$$

$$p(2i) = (2i)^2 - 4(2i) = -4 - 8i \neq 0$$

So,

$$w_p = \frac{-3e^{2it}}{-4 - 8i}$$

$$\Rightarrow w_p = \frac{-3e^{2it}}{-4 - 8i} \frac{-4 + 8i}{-4 + 8i} = \frac{(12 - 24i)e^{2it}}{80} = \frac{(3 - 6i)e^{2it}}{20}$$

$$w_p = \frac{3 - 6i}{20}(\cos 2t + i \sin 2t)$$

So then,

$$y_{p2} = \operatorname{Im}(w_p) = \frac{3}{20} \sin 2t - \frac{3}{10} \cos 2t$$

Approach 2: (without complex variables)

$$y_{p2} = A \sin 2t + B \cos 2t$$

$$y'_{p2} = 2A \cos 2t - 2B \sin 2t$$

$$y''_{p2} = -4A \sin 2t - 4B \cos 2t$$

Substituting into the DE:

$$-4A \sin 2t - 4B \cos 2t - 8A \cos 2t + 8B \sin 2t = -3 \sin 2t$$

We get two equations

$$-4A + 8B = -3$$

$$-8A - 4B = 0$$

Multiplying the second equation by 2 and adding to the first equation gives us:

$$-20A = -3 \Rightarrow A = \frac{3}{20}$$

Substituting,

$$-\frac{24}{20} = 4B \Rightarrow B = -\frac{24}{80} = -\frac{3}{10}$$

So,

$$y_{p2} = \frac{3}{20} \sin 2t - \frac{3}{10} \cos 2t$$

Therefore,

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$$y_p = -\frac{1}{8}t^2 - \frac{1}{16}t + \frac{3}{20}\sin 2t - \frac{3}{10}\cos 2t$$

and the general solution is:

$$y_g = c_1 + c_2 e^{4t} - \frac{1}{8}t^2 - \frac{1}{16}t + \frac{3}{20}\sin 2t - \frac{3}{10}\cos 2t$$

2(b) (13 pts.) Find a general solution of

$$x^2y'' + xy' - 9y = \frac{4}{x^3}$$

Solution: First solving the corresponding homogeneous DE:

$$x^2y'' + xy' - 9y = 0$$

This is Euler's equation. The indicial equation is:

$$p(m) = m^2 - 9 = 0$$

$$\Rightarrow m = -3, 3$$

$$y_h = c_1 x^3 + c_2 x^{-3}$$

We will use Variation of Parameters:

Let $y_1 = x^3$ and $y_2 = x^{-3}$. Then

$$y_p = v_1 x^3 + v_2 x^{-3}$$

We have the following two equations for v'_1 and v'_2 .

$$v'_1 x^3 + v'_2 x^{-3} = 0$$

$$3v'_1 x^2 - 3v'_2 x^{-4} = \frac{f}{a} = \frac{\frac{4}{x^3}}{x^2} = \frac{4}{x^5}$$

The Wronskian with $y_1 = x^3$ and $y_2 = x^{-3}$ is:

$$W[y_1, y_2] = y_1 y'_2 - y_2 y'_1 = x^3(-3x^{-4}) - x^{-3}(3x^2) = -6x^{-1}$$

Then

$$v'_1 = \frac{\begin{vmatrix} 0 & x^{-3} \\ \frac{4}{x^5} & -3x^{-4} \end{vmatrix}}{W[y_1, y_2]} = \frac{-\frac{4}{x^8}}{-6x^{-1}} = \frac{2}{3}x^{-7}$$

$$v'_2 = \frac{\begin{vmatrix} x^3 & 0 \\ 3x^2 & \frac{4}{x^5} \end{vmatrix}}{W[y_1, y_2]} = \frac{\frac{4}{x^2}}{-6x^{-1}} = -\frac{2}{3x}$$

Or using the formulas in the book,

$$v_1 = -\int \frac{y_2 f(x)}{W[y_1, y_2] a(x)} dx = -\int \frac{x^{-3}(4x^{-3})}{-6x^{-1}x^2} dx = \int \frac{2}{3}x^{-7} dx = -\frac{1}{9}x^{-6}$$

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$$v_2 = \int \frac{y_1 f(x)}{W[y_1, y_2] a(x)} dx = \int \frac{x^3 (4x^{-3})}{-6x^{-1} x^2} dx = \int -\frac{2}{3x} dx = -\frac{2}{3} \ln x$$

Therefore,

$$y_p = -\frac{1}{9}x^{-3} - \frac{2}{3}x^{-3} \ln x$$

Note that since x^{-3} is a homogeneous solution, we need not include $-\frac{1}{9}x^{-3}$ in the particular solution, although doing so is not wrong.

The general solution is:

$$y_g = c_1 x^3 + c_2 x^{-3} - \frac{1}{9}x^{-3} - \frac{2}{3}x^{-3} \ln x$$

or just

$$y_g = c_1 x^3 + c_2 x^{-3} - \frac{2}{3}x^{-3} \ln x$$

SNB check $x^2 y'' + xy' - 9y = \frac{4}{x^3}$, Exact solution is: $\left\{ \frac{C_2}{x^3} - \frac{1}{9x^3}(6 \ln x + 1) + C_3 x^3 \right\}$

3. (a) (15 pts.) Find:

$$\mathcal{L}^{-1} \left\{ \frac{s+3}{(s^2+1)(s^2-1)} \right\}$$

Solution: Approach 1:

$$\hat{y} = \frac{s+3}{(s^2+1)(s^2-1)} = \frac{s+3}{(s+1)(s-1)(s^2+1)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1}$$

$$\Rightarrow s+3 = A(s-1)(s^2+1) + B(s+1)(s^2+1) + Cs(s+1)(s-1) + D(s+1)(s-1)$$

$$s=1 \Rightarrow$$

$$B=1$$

$$s=-1 \Rightarrow$$

$$A = -\frac{1}{2}$$

Thus

$$\frac{s+3}{(s+1)(s-1)(s^2+1)} = \frac{-\frac{1}{2}}{s+1} + \frac{1}{s-1} + \frac{\frac{3}{2}}{s^2+1}$$

$$s=0 \Rightarrow$$

$$\frac{3}{-1} = -\frac{1}{2} - 1 + D$$

so

$$D = -\frac{3}{2}$$

and

$$\frac{s+3}{(s+1)(s-1)(s^2+1)} = \frac{-\frac{1}{2}}{s+1} + \frac{1}{s-1} + \frac{\frac{3}{2}}{s^2+1}$$

$$s=2 \Rightarrow$$

$$\frac{5}{3(1)(5)} = -\frac{1}{6} + 1 + \frac{2C - \frac{3}{2}}{5}$$

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or multiplying by 30

$$10 = -5 + 30 + 12C - 9$$

so

$$C = -\frac{1}{2}$$

So then,

$$\hat{y} = -\frac{1}{2} \frac{1}{s+1} + \frac{1}{s-1} + \frac{-\frac{1}{2}s - \frac{3}{2}}{s^2 + 1}$$

Therefore,

$$y = \mathcal{L}^{-1}\{\hat{y}\} = -\frac{1}{2}e^{-t} + e^t - \frac{1}{2}\cos t - \frac{3}{2}\sin t$$

Approach 2 (using complex variables):

$$\hat{y} = \frac{s+3}{(s^2+1)(s^2-1)} = \frac{s+3}{(s+1)(s-1)(s+i)(s-i)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s+i} + \frac{D}{s-i}$$

$$\Rightarrow s+3 = A(s-1)(s+i)(s-i) + B(s+1)(s+i)(s-i) + C(s+1)(s-1)(s-i) + D(s+1)(s-1)(s+i)$$

$$s = 1 \Rightarrow$$

$$4 = 2(1+i)(1-i)B \Rightarrow B = \frac{2}{(1+i)(1-i)} = 1$$

$$s = -1 \Rightarrow$$

$$2 = -2(-1+i)(-1-i)A \Rightarrow A = \frac{-1}{(-1+i)(-1-i)} = -\frac{1}{2}$$

$$s = -i \Rightarrow$$

$$3-i = (1-i)(-1-i)(-2i)C \Rightarrow 3-i = 4iC$$

$$\Rightarrow C = \frac{3-i}{4i} = -\frac{1}{4} - \frac{3}{4}i$$

$$s = i \Rightarrow$$

$$3+i = (1+i)(-1+i)(2i)D \Rightarrow 3+i = -4iD$$

$$\Rightarrow D = \frac{3+i}{-4i} = -\frac{1}{4} + \frac{3}{4}i$$

So then,

$$\hat{y} = -\frac{1}{2} \frac{1}{s+1} + \frac{1}{s-1} + \left(-\frac{1}{4} - \frac{3}{4}i\right) \frac{1}{s+i} + \left(-\frac{1}{4} + \frac{3}{4}i\right) \frac{1}{s-i}$$

Therefore,

$$y = \mathcal{L}^{-1}\{\hat{y}\} = -\frac{1}{2}e^{-t} + e^t + \left(-\frac{1}{4} - \frac{3}{4}i\right)e^{-it} + \left(-\frac{1}{4} + \frac{3}{4}i\right)e^{it}$$

$$\Rightarrow y = -\frac{1}{2}e^{-t} + e^t + \left(-\frac{1}{4} - \frac{3}{4}i\right)(\cos t - i \sin t) + \left(-\frac{1}{4} + \frac{3}{4}i\right)(\cos t + i \sin t)$$

$$\Rightarrow y = -\frac{1}{2}e^{-t} + e^t - \frac{1}{4}\cos t + \frac{1}{4}i \sin t - \frac{3}{4}i \cos t - \frac{3}{4}\sin t - \frac{1}{4}\cos t - \frac{1}{4}i \sin t + \frac{3}{4}i \cos t - \frac{3}{4}\sin t$$

$$\Rightarrow y = -\frac{1}{2}e^{-t} + e^t - \frac{1}{2}\cos t - \frac{3}{2}\sin t$$

(b) (10 pts.) Solve using Laplace Transforms:

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$$y'' + 2y' + 5y = 0 \quad y(0) = 1, \quad y'(0) = 2$$

Solution: Let

$$\begin{aligned}\mathcal{L}\{y\} &= \hat{y} \\ \mathcal{L}\{y'\} &= s\hat{y} - 1 \\ \mathcal{L}\{y''\} &= s^2\hat{y} - s - 2\end{aligned}$$

Substituting into the DE:

$$\begin{aligned}s^2\hat{y} - s - 2 + 2s\hat{y} - 2 + 5\hat{y} &= 0 \\ \Rightarrow \hat{y} &= \frac{s+4}{s^2+2s+5} = \frac{s+4}{s^2+2s+1+4} \\ &= \frac{s+4}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} + \frac{3}{(s+1)^2+4}\end{aligned}$$

So then the solution is:

$$y = \mathcal{L}^{-1}\{\hat{y}\} = e^{-t} \cos 2t + \frac{3}{2}e^{-t} \sin 2t$$

4.) Assume that the problem

$$\text{PDE : } u_{tt} = u_{xx} + 2u_x; \quad 0 < x < \pi, \quad t > 0$$

$$\text{B.C. : } u(0, t) = 0, \quad u(\pi, t) = 0$$

has a solution of the form $u(x, t) = X(x)T(t)$.a.) (10 pts.) Use separation of variables to derive the following eigenvalue problem for $X(x)$:

$$X'' + 2X' + \lambda X = 0; \quad X(0) = 0, \quad X(\pi) = 0$$

Solution: $u(x, t) = X(x)T(t)$ so, $u_x = X'T$, $u_{xx} = X''T$, and $u_{tt} = XT''$

Substituting into the PDE gives

$$XT'' = X''T + 2X'T$$

Separating the variables yields:

$$\begin{aligned}\frac{T''}{T} &= \frac{X'' + 2X'}{X} = -\lambda \\ \Rightarrow X'' + 2X' + \lambda X &= 0\end{aligned}$$

From the B.C.:

$$u(0, t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(\pi, t) = X(\pi)T(t) = 0 \Rightarrow X(\pi) = 0$$

b.) (15 pts.) Find the eigenvalues λ and the corresponding eigenfunctions for the problem given above in part a.).

The characteristic equation is:

$$p(r) = r^2 + 2r + \lambda = 0$$

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$$\Rightarrow r = \frac{-2 \pm \sqrt{4 - 4\lambda}}{2} = -1 \pm \sqrt{1 - \lambda}$$

There are three cases:

Case 1: $1 - \lambda = 0$

$$\Rightarrow r = -1 \text{ is a repeated root.}$$

$$\Rightarrow X = c_1 e^{-x} + c_2 x e^{-x}$$

$$0 = X(0) = c_1$$

$$0 = X(\pi) = c_2 \pi e^{-\pi} \Rightarrow c_2 = 0$$

$$\Rightarrow X \equiv 0 \text{ (trivial)}$$

Therefore $\lambda = 1$ is not an eigenvalue.Case 2: $1 - \lambda > 0$, Let $1 - \lambda = k^2$ where $k \neq 0$.

$$r = -1 \pm k$$

$$\Rightarrow X = c_1 e^{(-1+k)x} + c_2 e^{(-1-k)x}$$

$$0 = X(0) = c_1 + c_2$$

$$\Rightarrow c_2 = -c_1$$

so

$$X = c_1 e^{(-1+k)x} - c_1 e^{(-1-k)x}$$

$$0 = X(\pi) = c_1 e^{(-1+k)\pi} - c_1 e^{(-1-k)\pi}$$

$$0 = c_1 [e^{(-1+k)\pi} - e^{(-1-k)\pi}]$$

$$\Rightarrow c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow X \equiv 0 \text{ (trivial)}$$

Therefore there are no eigenvalues if $\lambda < 1$.Case 3: $1 - \lambda < 0$. Let $1 - \lambda = -k^2$ where $k \neq 0$

$$\Rightarrow r = -1 \pm ki$$

$$\Rightarrow X = c_1 e^{-x} \cos kx + c_2 e^{-x} \sin kx$$

$$0 = X(0) = c_1$$

$$0 = X(\pi) = c_2 e^{-\pi} \sin k\pi$$

$$\sin k\pi = 0 \Rightarrow k\pi = n\pi \Rightarrow k = n, \quad n = 1, 2, 3, \dots$$

$$1 - \lambda_n = -n^2$$

$$\Rightarrow \text{The eigenvalues are: } \lambda_n = 1 + n^2 \text{ where } n = 1, 2, 3, \dots$$

$$\text{and the eigenfunctions are: } X_n = c_n e^{-x} \sin(nx)$$

5. (a) (15 pts.) Find the first five nonzero terms of the Fourier sine series for the function

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$$f(x) = \begin{cases} -2 & 0 \leq x \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x \leq \pi \end{cases}$$

Be sure to give the Fourier series with these terms in it.

Solution:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Here $L = \pi$ so

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

where

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} (-2) \sin(nx) dx + \int_{\frac{\pi}{2}}^{\pi} 0 \cdot \sin(nx) dx \right] = -\frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin(nx) dx \\ &= \frac{4}{n\pi} \cos nx \Big|_0^{\frac{\pi}{2}} = \frac{4}{n\pi} \left[\cos\left(\frac{n\pi}{2}\right) - 1 \right] \end{aligned}$$

Therefore

$$\begin{aligned} a_1 &= -\frac{4}{\pi} \\ a_2 &= -\frac{4}{\pi} \\ a_3 &= -\frac{4}{3\pi} \\ a_4 &= 0 \\ a_5 &= -\frac{4}{5\pi} \\ a_6 &= -\frac{8}{6\pi} = -\frac{4}{3\pi} \end{aligned}$$

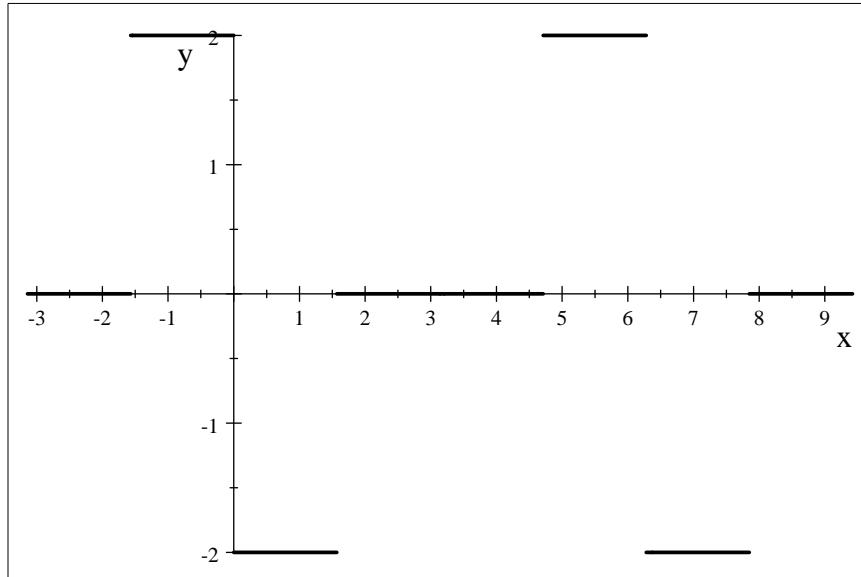
Thus

$$f(x) = -\frac{4}{\pi} \sin x - \frac{4}{\pi} \sin 2x - \frac{4}{3\pi} \sin 3x + 0 \sin 4x - \frac{4}{5\pi} \sin 5x - \frac{4}{3\pi} \sin 6x + \dots$$

(b) (10 pts.) Sketch the graph of the function represented by the Fourier sine series in 5 (a) on $-\pi \leq x \leq 3\pi$.

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6 (25 pts.)

$$\text{PDE} \quad u_{xx} = 9u_{tt}$$

$$\text{BCs} \quad u(0, t) = 0 \quad u_x(1, t) = 0$$

$$\text{ICs} \quad u(x, 0) = 0 \quad u_t(x, 0) = \frac{\pi}{6} \sin\left(\frac{7\pi x}{2}\right)$$

You must derive the solution. Your solution should not have any arbitrary constants in it. Show **all** steps.

Solution: We use separation of variables.

$$u(x, t) = X(x)T(t)$$

$$u_{xx}(x, t) = X''T$$

$$u_{tt}(x, t) = XT''$$

The PDE implies

$$X''T = 9XT''$$

or

$$\frac{X''}{X} = 9 \frac{T''}{T} = -\lambda^2 \quad \lambda \neq 0$$

so we have the two ODEs

$$X'' + \lambda^2 X = 0$$

$$T'' + \frac{1}{9}\lambda^2 T = 0$$

We have the boundary conditions

$$u(0, t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u_x(1, t) = X'(1)T(t) = 0 \Rightarrow X'(1) = 0$$

The solution of the ODE for X is

$$X(x) = a \sin \lambda x + b \cos \lambda x$$

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$$X(0) = b = 0 \text{ and } X(x) = a \sin \lambda x$$

$$X'(x) = a\lambda \cos \lambda x \text{ so } X'(1) = a\lambda \cos \lambda = 0$$

Thus

$$\lambda = (2n+1)\frac{\pi}{2} \quad n = 0, 1, 2, \dots$$

and

$$X_n(x) = A_n \sin\left(\frac{[2n+1]\pi x}{2}\right) \quad n = 0, 1, 2, \dots$$

The equation for $T(t)$ becomes

$$T'' + \left[(2n+1)\frac{\pi}{6}\right]^2 T = 0$$

so

$$T_n(t) = C_n \sin\left(\frac{[2n+1]\pi t}{6}\right) + D_n \cos\left(\frac{[2n+1]\pi t}{6}\right) \quad n = 0, 1, 2, \dots$$

Now

$$u(x, 0) = X(x)T(0) = 0 \Rightarrow T(0) = 0 \Rightarrow D_n = 0$$

and

$$T_n(t) = C_n \sin\left(\frac{[2n+1]\pi t}{6}\right) \quad n = 0, 1, 2, \dots$$

Thus

$$u_n(x, t) = X_n(x)T_n(t) = B_n \sin\left(\frac{[2n+1]\pi x}{2}\right) \sin\left(\frac{[2n+1]\pi t}{6}\right) \quad n = 0, 1, 2, \dots$$

Then

$$u(x, t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{[2n+1]\pi x}{2}\right) \sin\left(\frac{[2n+1]\pi t}{6}\right)$$

$$u_t(x, t) = \sum_{n=0}^{\infty} B_n \left(\frac{[2n+1]\pi}{6}\right) \sin\left(\frac{[2n+1]\pi x}{2}\right) \cos\left(\frac{[2n+1]\pi t}{6}\right)$$

and therefore

$$u_t(x, 0) = \sum_{n=0}^{\infty} B_n \left(\frac{[2n+1]\pi}{6}\right) \sin\left(\frac{[2n+1]\pi x}{2}\right) = \frac{\pi}{6} \sin\left(\frac{7\pi x}{2}\right)$$

Therefore

$$B_n = 0 \text{ for } n \neq 3$$

$$B_3 = \frac{1}{7}$$

and

$$u(x, t) = \frac{1}{7} \sin\left(\frac{7\pi x}{2}\right) \sin\left(\frac{7\pi t}{6}\right)$$

7. (a) (15 pts.) Find the power series solution to

$$y'' - x^2 y = 0$$

near $x = 0$. Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first six nonzero terms of the solution.

Solution:

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$$\begin{aligned}y(x) &= \sum_{n=0}^{\infty} a_n x^n \\y'(x) &= \sum_{n=1}^{\infty} a_n(n)x^{n-1} \\y''(x) &= \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}\end{aligned}$$

The DE implies

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

We shift the second series by letting $k-2 = n+2$ or $n = k-4$. Then since $k=4$ when $n=0$, we have

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - \sum_{k=4}^{\infty} a_{k-4} x^{k-2} = 0$$

or replacing k and n by m ,

$$a_2(2)(1)x^0 + a_3(3)(2)x + \sum_{m=4}^{\infty} (a_m(m)(m-1) - a_{m-4})x^{m-2}$$

Hence $a_2 = a_3 = 0$ and the recurrence relation is

$$a_m(m)(m-1) - a_{m-4} = 0 \quad m = 4, 5, 6, \dots$$

or

$$a_m = \frac{1}{m(m-1)} a_{m-4} \quad m = 4, 5, 6, \dots$$

Therefore

$$\begin{aligned}a_4 &= \frac{1}{4(3)} a_0 \\a_5 &= \frac{1}{5(4)} a_1 \\a_6 &= 0 \\a_7 &= 0 \\a_8 &= \frac{1}{8(7)} a_4 = \frac{1}{8(7)(4)(3)} a_0 \\a_9 &= \frac{1}{9(8)} a_5 = \frac{1}{9(8)(5)(4)} a_1\end{aligned}$$

so

$$y(x) = a_0 \left[1 + \frac{1}{4(3)} x^4 + \frac{1}{8(7)(4)(3)} x^8 + \dots \right] + a_1 \left[x + \frac{1}{5(4)} x^5 + \frac{1}{9(8)(5)(4)} x^9 + \dots \right]$$

(b) (10 pts.) Solve

$$y' + y = ty^3 \quad y(0) = 2$$

Solution: This is a Bernoulli equation. We rewrite the equation as

$$y'y^{-3} + y^{-2} = t$$

and let $v = y^{-2}$ so that $v' = -2y^{-3}y'$ and the above DE becomes

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$$v' - 2v = -2t$$

This is first order liner with $P = -2$ so $e^{\int P dt} = e^{-2t}$.

$$v'e^{-2t} - 2ve^{-2t} = \frac{d(ve^{-2t})}{dt} = -2te^{-2t}$$

so

$$ve^{-2t} = -(2) \frac{1}{4} (-2te^{-2t} - e^{-2t}) + C$$

Thus

$$v = \frac{1}{y^2} = \left(t + \frac{1}{2}\right) + Ce^{2t}$$

Since $y(0) = 2$

$$\frac{1}{4} = \frac{1}{2} + C \Rightarrow C = -\frac{1}{4}$$

and

$$\frac{1}{y^2} = \left(t + \frac{1}{2}\right) - \frac{1}{4}e^{2t}$$

8 (a) (15 pts.) The functions $y_1(t) = t$ and $y_2(t) = e^t$ are known to be solutions of

$$y'' + p(t)y' + q(t)y = 0$$

Determine the two functions $p(t)$ and $q(t)$.

Solution: Substituting y_1 and y_2 into the DE we have

$$\begin{aligned} p(t) + tq(t) &= 0 \\ e^t + p(t)e^t + q(t)e^t &= 0 \end{aligned}$$

or

$$\begin{aligned} p(t) + tq(t) &= 0 \\ p(t) + q(t) &= -1 \Rightarrow q(t) = -(p(t) + 1) \end{aligned}$$

so from the first equation

$$p(t) - tp(t) = t \Rightarrow p(t) = \frac{t}{1-t} \text{ and } q(t) = -\left(\frac{t}{1-t} + 1\right) = -\frac{1}{1-t}$$

The the DE is

$$y'' + \left(\frac{t}{1-t}\right)y' - \left(\frac{1}{1-t}\right)y = 0$$

(b) (10 pts.) Let $\{\Phi_n(x)\}$, $n = 1, 2, \dots$ be a set of orthonormal functions on $[a, b]$, that is,

$$\langle \Phi_n, \Phi_m \rangle = \int_a^b \Phi_n(x) \Phi_m(x) dx = \begin{cases} 0 & \text{for } n \neq m \\ 1 & \text{for } n = m \end{cases}$$

Let

$$f(x) = \sum_{n=1}^{\infty} a_n \Phi_n(x)$$

Show that

$$\sum_{i=1}^{\infty} [a_i]^2 = \int_a^b [f(x)]^2 dx$$

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$$\langle f(x), f(x) \rangle = \int_a^b [f(x)]^2 dx$$

However, we also have

$$\begin{aligned} \int_a^b [f(x)]^2 dx &= \langle f(x), f(x) \rangle = \left\langle \sum_{n=1}^{\infty} a_n \Phi_n(x), \sum_{n=1}^{\infty} a_n \Phi_n(x) \right\rangle = \left\langle a_1 \Phi_1 + a_2 \Phi_2 + \cdots, a_1 \Phi_1 + a_2 \Phi_2 + \cdots \right\rangle \\ &= \left\langle a_1 \Phi_1, a_1 \Phi_1 + a_2 \Phi_2 + \cdots \right\rangle + \left\langle a_2 \Phi_2, a_1 \Phi_1 + a_2 \Phi_2 + \cdots \right\rangle + \cdots \\ &= [a_1]^2 \langle \Phi_1, \Phi_1 \rangle + a_1 a_2 \langle \Phi_1, \Phi_2 \rangle + \cdots + a_2 a_1 \langle \Phi_2, \Phi_1 \rangle + [a_2]^2 \langle \Phi_2, \Phi_2 \rangle + \cdots \\ &= [a_1]^2 + [a_2]^2 + \cdots = \sum_{i=1}^{\infty} [a_i]^2 \end{aligned}$$

The last step comes from the fact that the set $\{\Phi_n(x)\}$, $n = 1, 2, \dots$ is orthonormal functions on $[a, b]$.

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > a$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > a$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} (\mathcal{L}\{f\}(s))$		

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2} \cos x \sin x + \frac{1}{2}x + C$
$\int x \cos bx dx = \frac{1}{b^2}(\cos bx + bx \sin bx) + C$
$\int x \sin bx dx = \frac{1}{b^2}(\sin bx - bx \cos bx) + C$
$\int \left(\frac{e^{-t}}{1+e^t} \right) dt = -e^{-t} - \ln(e^t) + \ln(1+e^t) + C$
$\int x e^{ax} dx = \frac{1}{a^2}(a x e^{ax} - e^{ax}) + C$