Name_________________________ Lecturer ______________________

Ma 221 Final Exam Solutions 5/08/07

Print Name: ________________________________

Lecture Section: _________

I pledge my honor that I have abided by the Stevens Honor System.

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This exam consists of 8 problems. You are to solve all of these problems. The point value of each problem is indicated. The total number of points is 200.

If you need more work space, continue the problem you are doing on the **other side of the page it is on**. Be sure that you do all problems.

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

There are tables giving Laplace transforms and integrals at the end of the exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

Total Score ________

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1. Solve

(a) (8 pts)

$$\left(2y^2x - 2y^3\right)dx + \left(4y^3 - 6y^2x + 2yx^2\right)dy = 0$$

Solution: $M = 2y^2x - 2y^3$ and $N = 4y^3 - 6y^2x + 2yx^2$. Therefore $M_y = 4yx - 6y^2 = N_x$ so this equation is exact. Thus, there exists $f(x,y)$ such that

$$f_x = M = 2y^2x - 2y^3 \text{ and } f_y = N = 4y^3 - 6y^2x + 2yx^2$$

Integrating the expression for f_x with respect to x, holding y fixed, yields

$$f = x^2y^2 - 2xy^3 + h(y)$$

Therefore

$$f_y = 2x^2y - 6xy^2 + h'(y) = N = 4y^3 - 6y^2x + 2yx^2$$

$$h'(y) = 4y^3 \Rightarrow h(y) = y^4 + C$$

Thus

$$f = x^2y^2 - 2xy^3 + y^4 + C$$

and the solution is given by

$$x^2y^2 - 2xy^3 + y^4 = k$$

(b) (7 pts) Solve

$$xy' - 2y = \frac{2}{3}x^5 \quad y(1) = \frac{2}{9}$$

Solution: We rewrite the DE as

$$y' - \frac{2}{x}y = \frac{2}{3}x^4$$

This is a first order linear DE. The integrating factor is $e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$. Multiplying the DE by this we get

$$\frac{1}{x^2}y' - \frac{2}{x^3}y = \frac{2}{3}x^2$$

or

$$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = \frac{2}{3}x^2$$

so

$$\frac{1}{x^2}y = \frac{2}{9}x^3 + C$$

or

$$y = \frac{2}{9}x^5 + Cx^2$$

The initial condition implies that

$$C = 0$$

so

$$y(x) = \frac{2}{9}x^5$$

1 (c) (10 pts) Find a general solution of

$$y'' - y' - 2y = e^{-5t} + 3e^{2t}$$

Solution: $p(r) = r^2 - r - 2 = (r-2)(r+1)$. So $r = -1, 2$ are roots of the characteristic equation. Therefore

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$$y_h = c_1e^{-t} + c_2e^{2t}$$

Since e^{-5t} is not a homogeneous solution we have

$$y_{p_1} = \frac{ke^{\alpha t}}{p(\alpha)} = \frac{e^{-5t}}{p(-5)} = \frac{e^{-5t}}{28}$$

Since e^{2t} is a homogeneous solution, but 2 is not a repeated root, we have

$$y_{p_2} = \frac{kte^{\alpha t}}{p'(\alpha)} = \frac{3te^{2t}}{p'(2)} = \frac{3te^{2t}}{3} = te^{2t}$$

Thus

$$y = y_h + y_{p_1} + y_{p_2} = c_1e^{-t} + c_2e^{2t} + \frac{e^{-5t}}{28} + te^{2t}$$

SNB check: $y'' - y' - 2y = e^{-5t} + 3e^{2t}$, Exact solution is:
$\left\{C_4e^{-t} + C_5e^{2t} + \frac{1}{84e^{5t}}(84te^{7t} - 28e^{7t} + 3)\right\}$

2. (a) (12 pts) Find a general solution of

$$y'' + 2y' - 3y = 5\sin 3t \ - 3 + 3t^2$$

Solution: $p(r) = r^2 + 2r - 3 = (r+3)(r-1),$ so $r = 1, -3.$ Thus

$$y_h = c_1e^t + c_2e^{-3t}$$

We find a particular solution for $5\sin 3t$ now.

Method 1 using complex variables.

Consider the companion equation

$$v'' + 2v' - 3v = 5\cos 3t$$

Multiplying this equation by i and adding it to the original equation and letting $w = iy + v$ leads to

$$w'' + 2w' - 3w = 5i\sin 3t + 5\cos 3t = 5e^{3it}$$

Since $p(3i) = -9 + 6i - 3 = -12 + 6i = -6(2-i) \neq 0$

$$w_p = -\frac{1}{6}\frac{5e^{3it}}{(2-i)} \times \frac{2+i}{2+i} = -\frac{5(2+i)}{6(5)}(\cos 3t + i\sin 3t)$$

Since $y_{p_1} = \operatorname{Im} w_{p_1}$

$$y_{p_1} = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t$$

We let $y_{p_2} = A_0 + A_1t + A_2t^2$ to find a particular solution for $-3 + 3t^2$. Then substituting into the DE leads to

$$2A_2 + 2A_1 + 4A_2t - 3A_0 - 3A_1t - 3A_2t^2 = -3 + 3t^2$$

Therefore

$$A_2 = -1$$

$$4A_2 - 3A_1 = 0 \Rightarrow A_1 = -\frac{4}{3}$$

$$2A_2 + 2A_1 - 3A_0 = -3$$

Thus

$$-2 - \frac{8}{3} - 3A_0 = -3 \Rightarrow A_0 = -\frac{5}{9}$$

$$y_{p_2} = -\frac{5}{9} - \frac{4}{3}t - t^2$$

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and finally

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^t + c_2 e^{-3t} - \frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t - \frac{5}{9} - \frac{4}{3}t - t^2$$

SNB check: $y'' + 2y' - 3y = 5\sin 3t - 3 + 3t^2$, Exact solution is:
$\left\{C_{47}e^{-3t} - \frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t - \frac{4}{3}t + C_{46}e^t - t^2 - \frac{5}{9}\right\}$

2(b) (13 pts.) Find a general solution of

$$y'' - 2y' + y = \frac{e^x}{x}$$

Solution: We use the Method of Variation of Parameters. The characteristic equation is $p(r) = r^2 - 2r + 1 = (r-1)^2$ so $r = 1$ is a repeated root and

$$y_h = c_1 e^x + c_2 x e^x$$

Let

$$y_p = v_1 e^x + v_2 x e^x$$

The two equations for v_1', v_2' are

$$v_1' e^x + v_2' x e^x = 0$$

$$v_1' e^x + v_2'(e^x + x e^x) = \frac{e^x}{x}$$

$$W[e^x, xe^x] = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

Therefore

$$v_1' = \frac{\begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & e^x + xe^x \end{vmatrix}}{e^{2x}} = -1 \Rightarrow v_1 = -x$$

$$v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix}}{e^{2x}} = \frac{1}{x} \Rightarrow v_2 = \ln x$$

Thus

$$y_p = v_1 e^x + v_2 x e^x = -xe^x + x \ln x e^x$$

Since xe^x is a homogeneous solution we need not include it in y_p and we have

$$y = y_h + y_p = c_1 e^x + c_2 x e^x + x \ln x e^x$$

SNB check: $y'' - 2y' + y = \frac{e^x}{x}$, Exact solution is: $\{C_{57}e^x - xe^x + C_{58}xe^x + xe^x \ln x\}$

3. (a) (10 pts.) Let

$$g(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1 \\ 0 & \text{for } t \ge 1 \end{cases}$$

Use the definition of the Laplace transform to find $\mathcal{L}\{g(t)\}$

Solution:

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$$\begin{aligned}\mathcal{L}\{g(t)\} &= \int_0^\infty g(t)e^{-st}dt \\ &= \int_0^1 1 \cdot e^{-st}dt + \int_1^\infty 0 \cdot e^{-st}dt \\ &= -\frac{1}{s}e^{-st}|_0^1 = -\frac{1}{s}[e^{-st} - 1] = \frac{1}{s} - \frac{e^{-st}}{s}\end{aligned}$$

(b) (15 pts.) Solve using Laplace Transforms:

$$y'' + 4y = 4t \qquad y(0) = 1,\ y'(0) = 5$$

Solution: Taking Laplace transforms of the DE yields

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} = \frac{4}{s^2}$$

or

$$\begin{aligned}\left(s^2+4\right)\mathcal{L}\{y\} &= s + 5 + \frac{4}{s^2} \\ \mathcal{L}\{y\} &= \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{4}{s^2\left(s^2+4\right)}\end{aligned}$$

We have to invert this to find $y(t)$.
Method 1 without complex variables.

$$\frac{4}{s^2\left(s^2+4\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

Multiplying by s^2 and setting $s = 0$ yields

$$\frac{4}{4} = 1 = B$$

so

$$\frac{4}{s^2\left(s^2+4\right)} = \frac{A}{s} + \frac{1}{s^2} + \frac{Cs+D}{s^2+4}$$

$s = 1$ gives the equation

$$\frac{4}{5} = A + 1 + \frac{C+D}{5}$$

$s = -1$ gives the equation

$$\frac{4}{5} = -A + 1 + \frac{-C+D}{5}$$

Adding these two equations gives

$$\frac{8}{5} = 2 + \frac{2}{5}D$$

or

$$8 = 10 + 2D$$

so $D = -1$. Thus

$$\frac{4}{s^2\left(s^2+4\right)} = \frac{A}{s} + \frac{1}{s^2} + \frac{Cs-1}{s^2+4}$$

$s = 2$ yields

$$\frac{4}{4(8)} = \frac{A}{2} + \frac{1}{4} + \frac{2C-1}{8}$$

or

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$$4 = 16A + 8 + 8C - 4$$

Simplifying we

$$2A + C = 0$$

The equation we got setting $s = 1$ becomes with $D = -1$

$$\frac{4}{5} = A + 1 + \frac{C-1}{5}$$

or

$$4 = 5A + 5 + C - 1$$

or

$$5A + C = 0$$

Thus $A = C = 0$ and we have

$$\frac{4}{s^2(s^2+4)} = \frac{1}{s^2} + \frac{-1}{s^2+4}$$

$$\begin{aligned}\mathcal{L}\{y\} &= \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{4}{s^2(s^2+4)} \\ &= \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{1}{s^2} + \frac{-1}{s^2+4} \\ &= \frac{s}{s^2+4} + \frac{4}{s^2+4} + \frac{1}{s^2}\end{aligned}$$

Taking the inverse of this expression using the tables yields

$$y(t) = \cos 2t + 2\sin 2t + x$$

Method 2 with complex variables:

$$\frac{4}{s^2(s^2+4)} = \frac{4}{s^2(s-2i)(s+2i)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2i} + \frac{D}{s+2i}$$

As before multiplying by s^2 and setting $s = 0$ yields $B = 1.$ Multiplying by $s - 2i$ and setting $s = 2i$ yields

$$\frac{4}{-4(4i)} = C = \frac{i}{4}$$

Multiplying by $s + 2i$ and setting $s = -2i$ yields

$$\frac{4}{-4(-4i)} = D = -\frac{i}{4}$$

Therefore

$$\frac{4}{s^2(s^2+4)} = \frac{4}{s^2(s-2i)(s+2i)} = \frac{A}{s} + \frac{1}{s^2} + \frac{i}{4}\frac{1}{s-2i} - \frac{i}{4}\frac{1}{s+2i}$$

$s = 1$ yields

$$\frac{4}{5} = A + 1 + \frac{i}{4(1-2i)} - \frac{i}{4}\frac{1}{1+2i}$$

or

$$-\frac{1}{5} = A + \frac{i}{4(1-2i)} - \frac{i}{4(1+2i)}$$
$$-\frac{1}{5} = A + \frac{i}{4(1-2i)} \times \frac{1+2i}{1+2i} + \frac{-i}{4(1+2i)} \times \frac{1-2i}{1-2i}$$
$$-\frac{1}{5} = A + \frac{i-2-i-2}{4(5)} = A - \frac{1}{5}$$

so $A = 0.$ Thus

$$\frac{4}{s^2\left(s^2+4\right)} = \frac{1}{s^2} + \frac{i}{4}\frac{1}{s-2i} - \frac{i}{4}\frac{1}{s+2i}$$

and

$$\begin{aligned}\mathcal{L}\{y\} &= \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{4}{s^2\left(s^2+4\right)} \\ &= \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{1}{s^2} + \frac{i}{4}\frac{1}{s-2i} - \frac{i}{4}\frac{1}{s+2i}\end{aligned}$$

so

$$\begin{aligned}y(t) &= \cos 2t + \frac{5}{2}\sin 2t + t + \frac{i}{4}e^{2it} - \frac{i}{4}e^{-2it} \\ &= \cos 2t + \frac{5}{2}\sin 2t + t + \frac{i}{4}\left[e^{2it} - e^{-2it}\right] \\ &= \cos 2t + \frac{5}{2}\sin 2t + t + \frac{i}{4}[\cos 2t + i\sin 2t - \cos 2t + i\sin 2t] \\ &= \cos 2t + \frac{5}{2}\sin 2t + t + \frac{i}{4}[2i\sin 2t] \\ &= \cos 2t + \frac{5}{2}\sin 2t + t - \frac{1}{2}\sin 2t \\ &= \cos 2t + 2\sin 2t + t\end{aligned}$$

4.) a.) (10 pts.) Use separation of variables, $u(x,t) = X(x)T(t),$ to find two ordinary differential equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$-12x^2t^5\frac{\partial^2 u}{\partial t^2} + (x+2)^3(t+2)^5\frac{\partial u}{\partial x} = 0.$$

Note: Do **not** solve these ordinary differential equations.
Solution:

$$\begin{aligned}\frac{\partial u}{\partial x} &= X'(x)T(t) \\ \frac{\partial u}{\partial t} &= X(x)T'(t) \\ \frac{\partial^2 u}{\partial t^2} &= X(x)T''(t)\end{aligned}$$

So the DE implies

$$-12x^2t^5X(x)T''(t) + (x+2)^3(t+2)^5X'(x)T(t) = 0$$

or

$$12x^2t^5X(x)T''(t) = (x+2)^3(t+2)^5X'(x)T(t)$$

So

$$\frac{12t^5T''}{(t+2)^5T} = \frac{(x+2)^3X'}{x^2X} = k$$

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So the two DEs are

$$12t^5T'' - k(t+2)^5T = 0$$
$$(x+2)^3X' - kx^2X = 0$$

b.) (15 pts.) Find the eigenvalues and eigenfunctions of

$$y'' - 4\lambda y' + 4\lambda^2 y = 0 \quad y(0) = 0 \;\; y(1) + y'(1) = 0$$

Solution: The characteristic equation is $r^2 - 4\lambda r + 4\lambda^2 = 0$ or $(r-2\lambda)^2 = 0.$ Thus $r = 2\lambda$ is a repeated root and the solution to the DE is

$$y(x) = c_1 e^{2\lambda x} + c_2 x e^{2\lambda x}$$

The boundary condition $y(0) = 0$ implies $c_1 = 0,$ so $y(x) = c_2 x e^{2\lambda x},$ and $y'(x) = c_2 e^{2\lambda x} + 2\lambda c_2 x e^{2\lambda x}.$ Therefore

$$y(1) + y'(1) = c_2 e^{2\lambda} + c_2 e^{2\lambda} + 2\lambda c_2 e^{2\lambda} = 0$$

or

$$(2+2\lambda)c_2 = 0$$

For $c_2 \neq 0$ this implies the eigenvalue $\lambda = -1$ and the eigenfunction

$$y(x) = c_2 x e^{-2x}$$

5. (a) (15 pts.) Find the first five nonzero terms of the Fourier *sine* series for the function

$$f(x) = \begin{cases} 2 & 0 \leq x \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < x \leq \frac{\pi}{2} \end{cases}$$

Be sure to give the Fourier series with these terms in it.
Solution: $L = \frac{\pi}{2}.$

$$f(x) = \sum_1^\infty a_n \sin\left(\frac{n\pi x}{L}\right) = \sum_1^\infty a_n \sin(2nx)$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)\sin(2nx)dx \quad n = 1,2,3,\ldots$$

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} (2)\sin(2nx)dx$$

$$= -\frac{8}{\pi}\,\frac{\cos 2nx}{2n}\bigg|_0^{\frac{\pi}{4}}$$

$$= -\frac{4}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \cos 0\right] = -\frac{4}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] \quad n = 1,2,3,\ldots$$

Therefore

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$$\begin{aligned}
a_1 &= -\frac{4}{\pi}\left[\cos\left(\frac{\pi}{2}\right) - 1\right] = \frac{4}{\pi} \\
a_2 &= -\frac{4}{2\pi}[\cos\pi - 1] = \frac{8}{2n} = \frac{4}{\pi} \\
a_3 &= -\frac{4}{3\pi}\left[\cos\left(\frac{3\pi}{2}\right) - 1\right] = \frac{4}{3\pi} \\
a_4 &= -\frac{4}{4\pi}[\cos(2\pi) - 1] = 0 \\
a_5 &= -\frac{4}{5\pi}\left[\cos\left(\frac{5\pi}{2}\right) - 1\right] = \frac{4}{5\pi} \\
a_6 &= -\frac{4}{6\pi}[\cos(3\pi) - 1] = \frac{4}{3\pi}
\end{aligned}$$

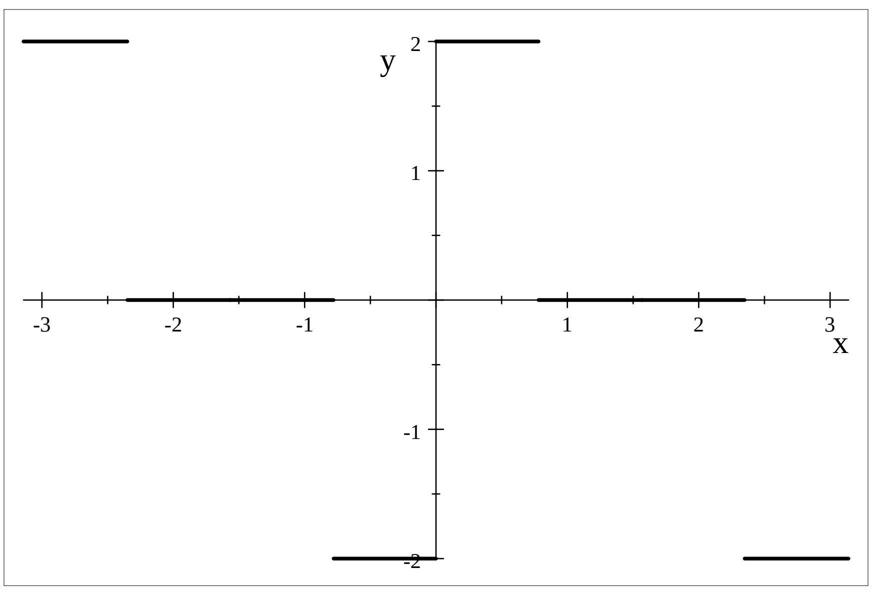
Thus

$$\begin{aligned}
f(x) &= \sum_{1}^{\infty} a_n \sin(2nx) = a_1 \sin 2x + a_2 \sin 4x + \cdots \\
&= \left(\frac{4}{\pi}\right)\sin 2x + \left(\frac{4}{\pi}\right)\sin 4x + \left(\frac{4}{3\pi}\right)\sin 6x + 0\sin 8x + \left(\frac{4}{5\pi}\right)\sin 10x + \left(\frac{4}{3\pi}\right)\sin 12x + \cdots
\end{aligned}$$

(b) (10 pts) Sketch the graph of the function represented by the Fourier sine series in 5 (a) on $-\pi \le x \le \pi$.

$\frac{\pi}{4} = 0.78540$, $\frac{\pi}{2} = 1.5708$ $\frac{3\pi}{4} = 2.3562$

$(0, 2, .78, 2)$



6 (25 pts.)

$$\begin{aligned}
&\text{PDE} \quad u_{xx} - 8u_t = 0 \\
&\text{BCs} \quad u(0,t) = 0 \qquad u_x(1,t) = 0 \\
&\text{ICs} \quad u(x,0) = -2\sin\frac{3\pi}{2}x + 10\sin\frac{9\pi}{2}x
\end{aligned}$$

You must derive the solution. Your solution should not have any arbitrary constants in it. Show **all**

steps.
Solution: Let $u(x,t) = X(x)T(t)$. Then the PDE implies

$$X''T = 8XT'$$

or

$$\frac{X''}{X} = 8\frac{T'}{T} = -r^2$$

Thus the ODEs for X and T are

$$X'' + r^2X = 0$$
$$T' + \frac{1}{8}r^2T = 0$$

We have the BCs $X(0) = X'(1) = 0$.

$$X(x) = a\sin rx + b\cos rx$$

$X(0) = 0$ implies $b = 0$.

$$X'(x) = ar\cos rx$$

so

$$X'(1) = ar\cos r = 0$$

Therefore $r = \frac{(2n+1)\pi}{2}, n = 0,1,2,\ldots$ and

$$X_n(x) = c_n\sin\left(\frac{2n+1}{2}\right)\pi x, \quad n = 0,1,2,\ldots$$

The equation for $T(t)$ is

$$T' + \frac{(2n+1)^2\pi^2}{32}T = 0$$

so

$$T_n(t) = d_ne^{-\frac{(2n+1)^2\pi^2}{32}t} \quad n = 0,1,2,\ldots$$

Hence

$$u_n(x,t) = A_n\sin\left(\frac{2n+1}{2}\right)\pi xe^{-\frac{(2n+1)^2\pi^2}{32}t} \quad n = 0,1,2,\ldots$$

$$u(x,t) = \sum_{n=0}^{\infty}u_n(x,t) = \sum_{n=0}^{\infty}A_n\sin\left[\left(\frac{2n+1}{2}\right)\pi x\right]e^{-\frac{(2n+1)^2\pi^2}{32}t}$$

$$u(x,0) = -2\sin\frac{3\pi}{2}x + 10\sin\frac{9\pi}{2}x = \sum_{n=0}^{\infty}A_n\sin\left(\frac{2n+1}{2}\right)\pi x$$

so $A_n = 0, n \neq 1,4$ and $A_1 = -2, A_4 = 10$.

$$u(x,t) = -2\sin\frac{3\pi}{2}xe^{-\frac{9\pi^2}{32}t} + 10\sin\frac{9\pi}{2}xe^{-\frac{81\pi^2}{32}t}$$

7. (a) (15 pts.) Solve the equation

$$y'' + 3xy' + 2y = 0$$

near $x = 0$. Be sure to give the recurrence relation and the first 3 nonzero terms in each linearly independent solution.
Solution:

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$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Then the DE implies

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} (3n+2) a_n x^n + 2a_0 = 0$$

Shifting the first summation by letting $n-2 = k,$ or $n = k+2$ we have

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{n=1}^{\infty} (3n+2) a_n x^n + 2a_0 = 0$$

Replacing k and n by m we have

$$\sum_{m=1}^{\infty} [(m+2)(m+1) a_{m+2} + (3m+2) a_m] x^m + 2(1) a_2 + 2a_0 = 0$$

Thus

$$a_2 = -a_0$$

and

$$(m+2)(m+1) a_{m+2} + (3m+2) a_m = 0$$

or

$$a_{m+2} = -\frac{3m+2}{(m+2)(m+1)} a_m \quad m = 1, 2, 3, \ldots$$

Thus

$$a_3 = -\frac{5}{3(2)} a_1$$
$$a_4 = -\frac{8}{(4)(3)} a_2 = \frac{8}{(4)(3)} a_0$$
$$a_5 = -\frac{11}{5(4)} a_3 = \frac{11(5)}{5(4)(3)(2)} a_1$$

Thus

$$y(x) = a_0\left(1 - x^2 + \frac{2}{3}x^4 + \cdots\right) + a_1\left(x - \frac{5}{3!}x^3 + \frac{55}{5!}x^5 + \cdots\right)$$

SNB check: $y'' + 3xy' + 2y = 0$, Series solution is:
$\Big\{ y(0) + xy'(0) - x^2 y(0) - \frac{5}{6}x^3 y'(0) + \frac{2}{3}x^4 y(0) + \frac{11}{24}x^5 y'(0) - \frac{14}{45}x^6 y(0) - \frac{187}{1008}x^7 y'(0) + \frac{1}{9}x^8 y(0) +$

(b) (10 pts.) Find a second order homogeneous, ordinary differential equation with real constant coefficients for which

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$$-5.2e^{-3t}\cos 5t \text{ and } \frac{2}{3}e^{-3t}\sin 5t$$

are the solutions.

Solution: The LI solutions are $e^{-3t}\cos 5t$ and $e^{-3t}\sin 5t$. These come from the case of complex roots with $\alpha = -3, \beta = 5.$ That is with roots $-3 \pm 5i.$ Thus

$$p(r) = (r+3-5i)(r-3+5i) = [(r+3)-5i][(r-3)+5i]$$
$$= (r+3)^2 + 25 = r^2 + 6r + 34$$

Therefore, the DE is

$$y'' - 6y' + 34y = 0$$

SNB check: $y'' + 6y' + 34y = 0$, Exact solution is: $\left\{C_5(\cos 5t)e^{-3t} - C_6 e^{-3t}\sin 5t\right\}$

8 (a) (12 pts.) Find an integrating factor to make

$$y^2\cos x dx + (4 + 5y\sin x)dy = 0$$

exact. Then use it to solve the equation.

Solution: We multiply the equation by $u(x,y)$ and get

$$y^2\cos x u dx + u(4 + 5y\sin x)dy = 0$$

Thus $M = y^2\cos x u$ and $N = u(4 + 5y\sin x)$.

$$M_y = 2yu\cos x + u_y y^2\cos x = N_x = u_x(4 + 5y\sin x)) + u(5y\cos x)$$

Letting $u_x = 0$ we get

$$2yu + y^2\frac{du}{dy} = 5yu$$

or

$$\frac{du}{dy} - \frac{3}{y}u = 0$$

The solution is $e^{-\int P dy} = e^{\int \frac{3}{y}dy} = y^3$ We multiply the original equation by this to get

$$y^5\cos x dx + \left(4y^3 + 5y^4\sin x\right)dy = 0$$

Now $M_y = 5y^4\cos x = N_x$ and this equation is exact. Thus there exists $f(x,y)$ such that

$$f_x = M = y^5\cos x \text{ and } f_y = N = 4y^3 + 5y^4\sin x$$

Therefore integrating f_x with respect to $x,$ holding y fixed gives

$$f(x,y) = y^5\sin x + h(y)$$

Hence

$$f_y = 5y^4\sin x + h'(y) = 4y^3 + 5y^4\sin x$$

so $h(y) = y^4 + c.$

$$f(x,y) = y^5\sin x + y^4 + c$$

and the solution is given by

$$y^5\sin x + y^4 = k$$

(b) (13 pts.) Find

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2 + 4s + 13}\right\}$$

Solution:

$$\frac{s+1}{s^2+4s+13} = \frac{s+1}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} - \frac{1}{(s+2)^2+9}$$
$$= \frac{s+2}{(s+2)^2+9} - \frac{1}{3}\frac{3}{(s+2)^2+9}$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+9}\right\}$$
$$= e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t$$

Name__________________________ Lecturer _______________________

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int x \cos bx dx = \frac{1}{b^2}(\cos bx + bx \sin bx) + C$
$\int x \sin bx = \frac{1}{b^2}(\sin bx - bx \cos bx) + C$
$\int \left(\frac{e^{-t}}{1+e^t}\right) dt = -e^{-t} - \ln(e^t) + \ln(1+e^t) + C$
$\int x e^{ax} dx = \frac{1}{a^2}(axe^{ax} - e^{ax}) + C$