Name_____ Lecturer _____

Ma 221 Final Exam Solutions 5/5/08

Print Name: _____

Lecture Section: _____

I pledge my honor that I have abided by the Stevens Honor System.

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This exam consists of 8 problems. You are to solve all of these problems. The point value of each problem is indicated. The total number of points is 200.

If you need more work space, continue the problem you are doing on the **other side of the page it is on**. Be sure that you do all problems.

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

There are tables giving Laplace transforms and integrals at the end of the exam.

Score on Problem #1 _____

#2 _____

#3 _____

#4 _____

#5 _____

#6 _____

#7 _____

#8 _____

Total Score _____

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1. Solve

(a) (8 pts)

$$\left(x^2+1\right)y' + 3xy = 6x \quad y(0) = -1$$

Solution: This equation if first order linear.

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

$e^{\int P dx} = e^{\frac{3}{2}\int \frac{2x}{x^2+1}dx} = e^{\frac{3}{2}\ln(x^2+1)} = \left(x^2+1\right)^{\frac{3}{2}}$. Multiplying the DE by this yields

$$\left(x^2+1\right)^{\frac{3}{2}}y' + 3x\left(x^2+1\right)^{\frac{1}{2}}y = 6x\left(x^2+1\right)^{\frac{1}{2}}$$

or

$$\frac{d}{dx}\left[\left(x^2+1\right)^{\frac{3}{2}}y\right] = 6x\left(x^2+1\right)^{\frac{1}{2}}$$

Integrating yields

$$\left(x^2+1\right)^{\frac{3}{2}}y = 2\left(x^2+1\right)^{\frac{3}{2}} + C$$

or

$$y(x) = 2 + \frac{C}{\left(x^2+1\right)^{\frac{3}{2}}}$$

The initial condition implies

$$y(0) = 2 + C = -1$$

so $C = -3$ and

$$y(x) = 2 - \frac{3}{\left(x^2+1\right)^{\frac{3}{2}}}$$

SNB check: $\begin{array}{c} y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1} \\ y(0) = -1 \end{array}$, Exact solution is: $\left\{\frac{1}{(x^2+1)^{\frac{3}{2}}}\left(2\left(x^2+1\right)^{\frac{3}{2}} - 3\right)\right\}$

(b) (7 pts) Solve

$$\sin(x+y)dx + (2y + \sin(x+y))dy = 0$$

Solution: This equation is exact since

$$\frac{\partial \sin(x+y)}{\partial y} = \cos(x+y) = \frac{\partial(2y+\sin(x+y))}{\partial x}$$

Thus, there exist $f(x,y)$ such that

$$f_x = M = \sin(x+y)$$
$$f_y = N = 2y + \sin(x+y)$$

Integrating the first equation wrt x we have

$$f(x,y) = -\cos(x+y) + g(y)$$

Then

$$f_y = \sin(x+y) + g'(y) = N = 2y + \sin(x+y)$$

and

$$g(y) = y^2 + C$$

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$$f(x,y) = -\cos(x+y) + y^2 + C$$

and the solution is given by

$$y^2 - \cos(x+y) = K$$

1 (c) (10 pts) Find a general solution of

$$y'' + 2y' - 3y = 3t + 4 + e^{-3t}$$

Solution: $p(r) = r^2 + 2r - 3 = (r+3)(r-1)$. Thus $r = -3, 1$ and

$$y_h = c_1 e^{-3t} + c_2 e^t$$

We must now find a particular solution. For e^{-3t} we have since $p(-3) = 0$ and $p'(-3) = 2(-3) + 2 = -4 \neq 0$

$$y_{p_1} = \frac{te^{-3t}}{p'(-3)} = \frac{te^{-3t}}{-4}$$

For $3t + 4$ we assume $y_{p_2} = At + B$, so $y' = A, y'' = 0$ and the DE implies

$$2A - 3At - 3B = 3t + 4$$

Hence $A = -1$ and $2A - 3B = 4$ implies $-3B = 6$ and $B = -2$ so

$$y_{p_2} = -t - 2$$

and

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^{-3t} + c_2 e^t - \frac{te^{-3t}}{4} - t - 2$$

SNB check: $y'' + 2y' - 3y = 3t + 4 + e^{-3t}$, Exact solution is:
$\left\{C_5 e^{-3t} - \frac{1}{16e^{3t}}\left(4t + 16te^{3t} + 32e^{3t} + 1\right) + C_4 e^t\right\}$

2. (a) (12 pts) Find a general solution of

$$y'' + 4y = \cos 2t - 2\sin t$$

Solution: The roots of the characteristic polynomial $p(r) = r^2 + 4$ are $r = \pm 2i$. Therefore

$$y_h = c_1 \sin 2t + c_2 \cos 2t$$

We now find a particular solution for $\cos 2t$. We use the Method of Undetermined Coefficients. Consider the two equations

$$y'' + 4y = \cos 2t$$
$$v'' + 4v = \sin 2t$$

Multiplying the second equation by i and adding it to the first and letting $w = y + iv$ leads to

$$w'' + 4w = \cos 2t + i\sin 2t = e^{2it}$$

Thus

$$w_p = \frac{e^{2it}}{p'(2i)} = \frac{te^{2it}}{2(2i)} = -\frac{ite^{2it}}{4} = \frac{t}{4}(-i\cos 2t + \sin 2t)$$

Thus

$$y_{p_1} = \operatorname{Re} w_p = \frac{t\sin 2t}{4}$$

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To find a particular solution for $-2\sin t$ we consider the two equations

$$y'' + 4y = -2\sin t$$
$$v'' + 4v = -2\cos t$$

Multiply the first equation by i, let $u = v + iy$ to get

$$u'' + 4u = -2\cos t - 2\sin t = -2e^{it}$$

Then since $p(i) = i^2 + 4 = 3 \neq 0$, then

$$u_p = \frac{-2e^{it}}{3} = -\frac{2}{3}(\cos t + i\sin t)$$

and

$$y_{p_2} = \operatorname{Im} u_p = -\frac{2}{3}\sin t$$

Thus

$$y = y_h + y_{p_1} + y_{p_2} = c_1 \sin 2t + c_2 \cos 2t + \frac{t\sin 2t}{4} - \frac{2}{3}\sin t$$

SNB check: $y'' + 4y = \cos 2t - 2\sin t$, Exact solution is:
$\left\{\frac{1}{16}\cos 2t - \frac{2}{3}\sin t + C_{16}\cos 2t - C_{17}\sin 2t + \frac{1}{4}t\sin 2t\right\}$

2(b) (13 pts.) Use the method of Variation of Parameters to find a general solution of

$$y'' + 2y' + y = t^5 e^{-t}$$

Solution: The characteristic equation is $p(r) = r^2 + 2r + 1 = (r+1)^2$, so $r = -1$ is a repeated root and

$$y_h = c_1 e^{-t} + c_2 e^{-t}$$

Let

$$y_p = v_1 e^{-t} + v_2 t e^{-t}$$

The two equations for v_1' and v_2' are

$$e^{-t}v_1' + te^{-t}v_2' = 0$$
$$-e^{-t}v_1' + (e^{-t} - te^{-t})v_2' = t^5 e^{-t}$$

The Wronskian of these the two homorgeneous solutions is

$$\begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t}$$

Thus

$$v_1' = \frac{\begin{vmatrix} 0 & te^{-t} \\ t^5 e^{-t} & e^{-t} - te^{-t} \end{vmatrix}}{e^{-2t}} = \frac{-t^6 e^{-2t}}{e^{-2t}} = -t^6$$

$$v_2' = \frac{\begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & t^5 e^{-t} \end{vmatrix}}{e^{-2t}} = t^5$$

Thus

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$$v_1 = -\frac{t^7}{7}$$
$$v_2 = \frac{t^6}{6}$$
$$y_p = v_1 e^{-t} + v_2 t e^{-t} = -\frac{t^7}{7}e^{-t} + \frac{t^7}{6}e^{-t} = \frac{1}{42}t^7 e^{-t}$$

and

$$y = y_h + y_p = c_1 e^{-t} + c_2 e^{-t} + \frac{1}{42}t^7 e^{-t}$$

SNB check $y'' + 2y' + y = t^5 e^{-t}$, Exact solution is: $\left\{C_{25}e^{-t} + \frac{1}{42}\frac{t^7}{e^t} + C_{26}te^{-t}\right\}$

3. (a) (10 pts.) Let

$$g(t) = \begin{cases} t & \text{for } 0 \le t \le 1 \\ 1 & \text{for } 1 < t < \infty \end{cases}$$

Use the definition of the Laplace transform to find $\mathcal{L}\{g(t)\}$

Solution:

$$\begin{aligned}\mathcal{L}\{g(t)\} &= \int_0^\infty g(t)e^{-st}dt = \int_0^1 te^{-st}dt + \int_1^\infty (1)e^{-st}dt \\ &= \frac{1}{s^2}(-ste^{-st} - e^{-st})]_0^1 - \frac{1}{s}e^{-st}]_1^\infty \\ &= \frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-s}}{s} = \frac{1}{s^2} - \frac{e^{-s}}{s^2}\end{aligned}$$

The formula $\int te^{at}dt = \frac{1}{a^2}(ate^{at} - e^{at}) + C$ on the last page of the exam was used in the calculation.

(b) (15 pts.) Solve using Laplace Transforms:

$$y'' - y' - 2y = e^{2t} \qquad y(0) = 0,\ y'(0) = -1$$

Solution: Taking Laplace transforms of both sides of the equation leads to

$$\left(s^2 - s - 2\right)\mathcal{L}\{y\} - sy(0) - y'(0) + y(0) = \mathcal{L}\left\{e^{2t}\right\} = \frac{1}{s-2}$$

Thus

$$(s-2)(s+1)\mathcal{L}\{y\} = -1 + \frac{1}{s-2} = \frac{-s+3}{s-2}$$

so that

$$\mathcal{L}\{y\} = \frac{-s+3}{(s+1)(s-2)^2}$$

We have to invert $\frac{-s+3}{(s+1)(s-2)^2}$. Now

$$\frac{-s+3}{(s+1)(s-2)^2} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

Multiplying by $s+1$ and setting $s = -1$ gives us $A = \frac{4}{(-3)^2} = \frac{4}{9}$. Multiplying by $(s-2)^2$ and setting $s = 2$ gives us $C = \frac{1}{3}$. Thus

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$$\frac{-s+3}{(s+1)(s-2)^2} = \frac{4}{9}\frac{1}{s+1} + \frac{B}{s-2} + \frac{1}{3}\frac{1}{(s-2)^2}$$

Setting $s = 1$ we get $\frac{2}{2} = \frac{2}{9} - B + \frac{1}{3}$. Thus

$$B = -1 + \frac{5}{9} = -\frac{4}{9}$$

$$\frac{-s+3}{(s+1)(s-2)^2} = \frac{4}{9}\frac{1}{s+1} - \frac{4}{9}\frac{1}{s-2} + \frac{1}{3}\frac{1}{(s-2)^2}$$

$$y(t) = \mathcal{L}\left\{\frac{4}{9}\frac{1}{s+1} - \frac{4}{9}\frac{1}{s-2} + \frac{1}{3}\frac{1}{(s-2)^2}\right\} = \frac{4}{9}e^{-t} - \frac{4}{9}e^{2t} + \frac{1}{3}te^{2t}$$

where we have used the shift formula to get the last inverse.
SNB check:

$$\begin{aligned} y'' - y' - 2y &= e^{2t} \\ y(0) &= 0 \\ y'(0) &= -1 \end{aligned}$$

, Exact solution is: $\left\{\frac{4}{9}e^{-t} - \frac{1}{3}e^{2t} + \frac{1}{3}te^{2t} - \frac{1}{9}e^{2t}\right\}$,

4.) a.) (10 pts.) Use separation of variables, $u(x,t) = X(x)T(t)$, to find two ordinary differential equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$-3x^4t^3\frac{\partial^2 u}{\partial x^2} + (x+6)^5(t-2)^3\frac{\partial u}{\partial t} = 0.$$

Note: Do **not** solve these ordinary differential equations.
Solution:

$$\begin{aligned} \frac{\partial u}{\partial x} &= X'(x)T(t) \\ \frac{\partial u}{\partial x} &= X''(x)T(t) \\ \frac{\partial u}{\partial t} &= X(x)T'(t) \end{aligned}$$

So the DE implies

$$-3x^4t^3X''T + (x+6)^5(t-2)^3XT' = 0$$

or

$$\frac{3x^4X''}{(x+6)^5X} = \frac{(t-2)^3T'}{t^3T} = k$$

where k is a constant. The two DEs are

$$\begin{aligned} 3x^4X'' - k(x+6)^5X &= 0 \\ (t-2)^3T' - kt^3T &= 0 \end{aligned}$$

b.) (15 pts.) Find the eigenvalues and eigenfunctions for

$$x^2y'' + xy' + \lambda y = 0 \qquad y(1) = y(e^{\pi}) = 0$$

Be sure to consider the cases $\lambda < 0, \lambda = 0,$ and $\lambda > 0$.

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Solution:

Note that this is an Euler equation!! Since $p = 1, q = \lambda$ the indicial equation is

$$r^2 + (p-1)r + q = r^2 - \lambda = 0$$

I. $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. Then the indicial equation is

$$r^2 - \alpha^2 = 0$$

so $m = \pm\alpha$. Thus

$$y = c_1 x^{\alpha} + c_2 x^{-\alpha}$$

The boundary conditions imply

$$c_1 + c_2 = 0$$
$$c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = 0$$

This leads to $c_1 = c_2 = 0$, so $y = 0$ and there are no negative eigenvalues.

II. $\lambda = 0$. The DE for this case is

$$x^2 y'' + xy' = 0$$

or

$$xy'' + y' = 0$$

Let $v = y'$ so we have

$$xv' + v = 0$$
$$(xv)' = 0$$

Then $v = \frac{c_1}{x}$, and $y = c_1 \ln x + c_2$. The BCs lead to

$$c_1 + c_2 = 0$$
$$c_1 \ln e^{\pi} + c_2 = 0$$

or

$$c_1 + c_2 = 0$$
$$\pi c_1 + c_2 = 0$$

Again we have $c_1 = c_2 = 0$, so $\lambda = 0$ is not an eigenvalue.

III. $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The indicial equation is

$$r^2 + \beta^2 = 0$$

so

$$r = \pm\beta i$$

Thus

$$y = c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)$$

The BCs imply

$$c_1 = 0$$
$$c_2 \text{ is arbitrary}$$
$$\sin(\beta\pi) = 0 \Rightarrow \beta = n \text{ for } n = 1, 2, \ldots$$

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Eigenvalues:

$$\lambda_n = \beta^2 = n^2 \text{ where } n = 1, 2, \ldots$$

Eigenfunctions:

$$y_n = c_n \sin(n \ln x)$$

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5. (a) (15 pts.) Find the first five nonzero terms of the Fourier *cosine* series for the function

$$f(x) = 2 - x; \quad 0 < x < 2$$

Be sure to give the Fourier series with these terms in it.
Solution:

$$f(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$$

$$b_0 = \frac{1}{L}\int_0^L f(x)dx \quad b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \;\; n = 1, 2, \ldots$$

Here $L = 2$ so

$$f(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{2}\right)$$

$$b_0 = \frac{1}{2}\int_0^2 f(x)dx = \frac{1}{2}\int_0^2 (2-x)dx = \frac{1}{2}(2x - \frac{x^2}{2})\Big|_0^2 = 1$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{2}\int_0^2 (2-x)\cos\frac{n\pi x}{2}dx \;\; n = 1, 2, \ldots$$

so

$$b_n = [\frac{2}{n\pi}(2-x)\sin(\frac{n\pi x}{2}) - \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)]_0^2 = \left(\frac{4}{n^2\pi^2}\right)[-\cos(n\pi) + 1] \quad n = 1, 2, \ldots$$

Thus

$$b_1 = \left(\frac{4}{\pi^2}\right)(1+1) = \frac{8}{\pi^2}$$

$$b_2 = \left(\frac{4}{4\pi^2}\right)[-1+1] = 0$$

$$b_3 = \left(\frac{4}{9\pi^2}\right)[1+1] = \frac{8}{9\pi^2}$$

$$b_4 = \left(\frac{4}{16\pi^2}\right)[-1+1] = 0$$

$$b_5 = \left(\frac{4}{25\pi^2}\right)[1+1] = \frac{8}{25\pi^2}$$

$$b_6 = \left(\frac{4}{36\pi^2}\right)[-1+1] = 0$$

$$b_7 = \left(\frac{4}{49\pi^2}\right)[1+1] = \frac{8}{49\pi^2}$$

and

$$f(x) = 1 + \frac{8}{\pi^2}\cos(\frac{\pi x}{2}) + \frac{8}{9\pi^2}\cos(\frac{3\pi x}{2}) + \frac{8}{25\pi^2}\cos(\frac{5\pi x}{2}) + \frac{8}{49\pi^2}\cos(\frac{7\pi x}{2}) + \cdots$$

(b) (10 pts.) Sketch the graph of the function represented by the Fourier cosine series in 5 (a) on $-2 < x < 6$.

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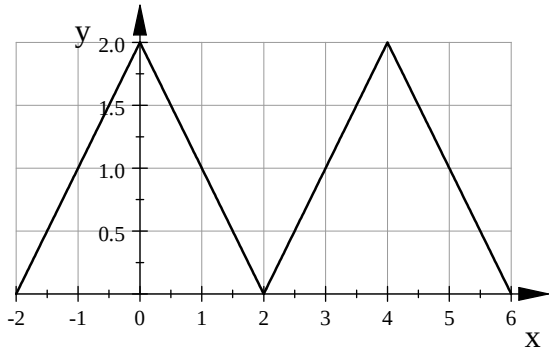

$2 - x$

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6 (25 pts)

$$\begin{aligned} \text{PDE} \quad & u_{tt} = 4u_{xx} \\ \text{BCs} \quad & u_x(0,t) = 0 \qquad u_x(\pi,t) = 0 \\ \text{ICs} \quad & u(x,0) = 2\cos(3x) - 5\cos(5x) + 7\cos(6x) \\ & u_t(x,0) = 0 \end{aligned}$$

You must derive the solution. Your solution should not have any arbitrary constants in it. Show **all** steps.

Solution:
Let $u(x,t) = X(x)T(t)$. Then differentiating and substituting in the PDE yields

$$XT'' = 4X''T$$
$$\Rightarrow$$
$$\frac{X''}{X} = \frac{T''}{4T}$$

Using the argument that the left hand side is purely a function of x and the right hand side is purely a function of t, and the only way that they can be equal is if they are equal to a constant, we get

$$\frac{X''}{X} = \frac{T''}{4T} = k \qquad k \text{ a constant}$$

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T'' - 4kT = 0$$

The boundary condition $u_x(0,t) = 0$ implies, since $u_x(x,t) = X'(x)T(t)$ that $X'(0)T(t) = 0$. We cannot have $T(t) = 0$, since this would imply that $u(x,t) = 0$. Thus $X'(0) = 0$. Similarly, the boundary condition $u_x(\pi,t) = 0$ leads to $X'(\pi) = 0$.

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X'(0) = X'(\pi) = 0$$

For $k > 0$, the only solution is $X = 0$. For $k = 0$ we have $X = Ax + B$. $X'(x) = A$, so the BCs imply that

$$X(x) = B, \quad B \neq 0$$

is a nontrivial solution corresponding to the eigenvalue $k = 0$.
For $k < 0$, let $-k = \alpha^2$, where $\alpha \neq 0$. Then we have the equation

$$X'' + \alpha^2 X = 0$$

and

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$$X(x) = c_1 \sin \alpha x + c_2 \cos \alpha x$$
$$X'(x) = c_1 \alpha \cos \alpha x - c_2 \alpha \sin \alpha x$$
$$X'(0) = c_1 \alpha = 0$$

so $c_1 = 0$.

$$X'(\pi) = -c_2 \alpha \sin \alpha \pi = 0$$

Therefore $\alpha = n,\ n = 1, 2, \ldots$ and the solution is

$$k = -n^2 \qquad X_n(x) = c_n \cos nx \qquad n = 1, 2, 3, \ldots$$

Substituting the values of $k = -n^2$ into the equation for $T(t)$ leads to

$$T'' + 4n^2 T = 0$$

which has the solution

$$T_n(t) = c_{n1} \cos 2nt + c_{n2} \sin 2nt, \quad n = 1, 2, 3, \ldots$$

$u_t(x, 0) = 0$ implies that $T'(0) = 0$.

$T'(t) = -2nc_{n1} \sin 2nt + 2nc_{n2} \cos 2nt$ so then $T'(0) = 0$ implies that $c_{n2} = 0$ for $n = 1, 2, 3, \ldots$

We now have the solutions

$$u_n(x, t) = A_n \cos nx \cos 2nt \qquad n = 0, 1, 2, 3, \ldots$$

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} A_n \cos nx \cos 2nt$$

satisfies the PDE, the boundary conditions, and the second initial condition.
The first initial condition leads to

$$u(x, 0) = 2\cos(3x) - 5\cos(5x) + 7\cos(6x) = \sum_{n=0}^{\infty} A_n \cos nx.$$

Matching the cosine terms on both sides of this equation leads to

$A_3 = 2,\ A_5 = -5,$ and $A_6 = 7.$ All of the other constants must be zero, since there are no cosine terms or constant terms on the left to match with. Thus

$$u(x, t) = 2\cos 3x \cos 6t - 5\cos 5x \cos 10t + 7\cos 6x \cos 12t$$

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7. (a) (10 pts.) Consider the equation

$$(x^2 - 8x + 15)y'' + (x + 2)y' + 3y = 0$$

i.) Find the singular points of the above equation

ii.) Find a **minimum value** for the radius of convergence of a power series solution about $x = 2$ of the above equation. (Note: You do NOT have to find the power series solution.)

Solution:

i.) Divide the DE by $x^2 - 8x + 15$ to obtain:

$$y'' + \frac{(x+2)}{x^2 - 8x + 15}y' + \frac{3}{x^2 - 8x + 15}y = 0$$

$x^2 - 8x + 15 = 0 \Rightarrow (x-3)(x-5) = 0 \Rightarrow x = 3, 5$

So then, the singular points are $x = 3$ and $x = 5$.

ii.) The minimum radius of convergence will be the distance from $x = 2$ to the nearest singular point.

The nearest singular point is $x = 3$, so then, the minimum radius of convergence is 1.

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(b) (15 pts.) Find the first 6 nonzero terms of each linearly independent power series solution about $x = 0$ for the DE:

$$y'' - x^3 y' + y = 0$$

Be sure to give the recurrence relation.

Solution:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

The DE implies

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=1}^{\infty} a_n (n) x^{n+2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

We shift the first sum by letting $n - 2 = k + 2$ or $n = k + 4$ and the third sum by letting $n = k + 2$and get

$$\sum_{k=-2}^{\infty} a_{k+4}(k+4)(k+3)x^{k+2} - \sum_{n=1}^{\infty} a_n (n) x^{n+2} + \sum_{k=-2}^{\infty} a_{k+2} x^{k+2} = 0$$

Replacing n and k by m and combining we have

$$2a_2 + 6a_3 x + 12a_4 x^2 + a_0 + a_1 x + a_2 x^2 + \sum_{m=1}^{\infty} \{a_{m+4}(m+4)(m+3) - m a_m + a_{m+2}\} x^{m+2} = 0$$

so

$$2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2}$$

$$6a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{6}$$

$$12a_4 + a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{12} = \frac{a_0}{24}$$

$$a_{m+4}(m+4)(m+3) - m a_m + a_{m+2} = 0$$

or

$$a_{m+4} = \frac{m a_m - a_{m+2}}{(m+4)(m+3)} \quad m = 3, 4, \ldots$$

Then

$$a_5 = \frac{a_1 - a_3}{(5)(4)} = \frac{7a_1}{120}$$

Thus

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots = a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots\right] + a_1\left[x - \frac{1}{6}x^3 + \frac{7}{120}x^5 + \cdots\right]$$

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8 (a) (12 pts.) Consider:

$$y'' + p(t)y' + q(t)y = 1$$

Given that $y_1(t) = t$ and $y_2(t) = t^2$ are solutions to the above equation, find $p(t)$ and $q(t)$.

Solution: Substituting y_1 and y_2 into the DE we have

$$p(t) + tq(t) = 1$$
$$2 + p(t)(2t) + q(t)t^2 = 1$$

or

$$p(t) + tq(t) = 1 \Rightarrow q(t) = \frac{1 - p(t)}{t}$$
$$2tp(t) + t^2 q(t) = -1$$

so from the second equation

$$2tp(t) + t(1 - p(t)) = -1 \quad \Rightarrow 2tp(t) - tp(t) = -1 - t \quad \Rightarrow p(t) = \frac{-1 - t}{t}$$

$$\text{and } q(t) = \frac{1 + \frac{1+t}{t}}{t} = \frac{2t + 1}{t^2}$$

The the DE is

$$y'' + \left(\frac{-1 - t}{t}\right)y' + \left(\frac{2t + 1}{t^2}\right)y = 1$$

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(b) (13 pts.) Find

$$\mathcal{L}^{-1}\left\{\frac{5s^2-5s-4}{(s-3)(s^2+4)}\right\}$$

Solution:

$$\hat{y} = \frac{5s^2-5s-4}{(s-3)(s^2+4)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow 5s^2-5s-4 = A(s^2+4)+Bs(s-3)+C(s-3)$$

$s = 3 \Rightarrow$

$$26 = 13A \Rightarrow A = 2$$

$s = 0 \Rightarrow$

$$-4 = 4A-3C \Rightarrow C = 4$$

$s = 1 \Rightarrow$

$$-4 = 5A-2B-2C \Rightarrow B = 3$$

So then,

$$\hat{y} = \frac{2}{s-3} + \frac{3s+4}{s^2+4} = \frac{2}{s-3} + 3\left(\frac{s}{s^2+4}\right) + 2\left(\frac{2}{s^2+4}\right)$$

Therefore,

$$y = \mathcal{L}^{-1}\{\hat{y}\} = 2e^{3t} + 3\cos 2t + 2\sin 2t$$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int x \cos bx dx = \frac{1}{b^2}(\cos bx + bx \sin bx) + C$
$\int x \sin bx = \frac{1}{b^2}(\sin bx - bx \cos bx) + C$
$\int \left(\frac{e^{-t}}{1+e^t}\right) dt = -e^{-t} - \ln(e^t) + \ln(1+e^t) + C$
$\int te^{at} dt = \frac{1}{a^2}(ate^{at} - e^{at}) + C$