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Ma 221 Final Exam Solutions 5/7/09

Print Name: ________________________________

Lecture Section: _________

I pledge my honor that I have abided by the Stevens Honor System.

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This exam consists of 8 problems. You are to solve all of these problems. The point value of each problem is indicated. The total number of points is 200.

If you need more work space, continue the problem you are doing on the **other side of the page it is on**. Be sure that you do all problems.

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

There are tables giving Laplace transforms and integrals at the end of the exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

Total Score ________

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1. Solve

(a) (8 pts)

$$x\cos x dx + (1 - 6y^5)dy = 0 \quad y(\pi) = 0$$

Solution: This equation is separable. Hence

$$\int x\cos x dx + \int (1 - 6y^5)dy = c$$

From the table below $\int x\cos x dx = \cos x + x\sin x + k$ so

$$\cos x + x\sin x + y - y^6 = C$$

The initial condition $y(\pi) = 0$ implies

$$-1 = C$$

so the solution is

$$\cos x + x\sin x + 1 = y^6 - y$$

(b) (7 pts) Solve

$$y' = \frac{2 + ye^{xy}}{2y - xe^{xy}}$$

Solution: The equation may be rewritten as

$$(2 + ye^{xy})dx + (xe^{xy} - 2y)dy = 0$$

Let $M = 2 + ye^{xy}$ and $N = xe^{xy} - 2y.$ Then

$$M_y = e^{xy} + xye^{xy} = N_x$$

and this equation is exact. There exists a function $f(x,y)$ such that

$$f_x = M = 2 + ye^{xy} \quad \text{and} \quad f_y = N = xe^{xy} - 2y$$

$\Rightarrow$

$$f = 2x + e^{xy} + g(y)$$

Then

$$f_y = xe^{xy} + g'(y) = N = xe^{xy} - 2y$$

Thus $g(y) = -y^2 + c.$ The solution is given by

$$f = 2x + e^{xy} - y^2 = k$$

1 (c) (10 pts) Find a general solution of

$$y'' - y' - 2y = 64e^{-t} + 4t^2$$

Solution: The characteristic polynomial is

$$p(r) = r^2 - r - 2 = (r - 2)(r + 1)$$

Therefore the roots are $r = 2, -1$ and

$$y_h = c_1e^{-t} + c_2e^{2t}$$

We first find a particular solution for $64e^{-t}.$ Since $p(-1) = 0$ and $p'(r) = 2r - 1$ so $p'(-1) = -3 \neq 0,$ then

$$y_{p_1} = \frac{64te^{-t}}{-3}$$

To find a particular solution for $4t^2$ we let

$$y_{p_2} = At^2 + Bt + C$$
$$y'_{p_2} = 2At + B$$
$$y''_{p_2} = 2A$$

Substituting into the DE we have

$$2A - 2At - B - 2At^2 - 2Bt - 2C = 4t^2$$

Thus

$$A = -2 \quad -A - B = 0 \quad 2A - B - 2C = 0$$

Hence $B = 2$ and $C = -3$ so

$$y_{p_2} = -2t^2 + 2t - 3$$

Finally

$$y_g = y_h + y_{p_1} + y_{p_2} = c_1 e^{-t} + c_2 e^{2t} - \frac{64te^{-t}}{3} - 2t^2 + 2t - 3$$

$y'' - y' - 2y = 64e^{-t} + 4t^2$, Exact solution is:
$\left\{C_8 e^{-t} - \frac{1}{9e^t}\left(192t + 27e^t + 18t^2e^t - 18te^t + 64\right) + C_9 e^{2t}\right\}$

2. (a) (12 pts) Find a general solution of

$$y'' - 2y' + y = 4\sin x$$

Solution: The characteristic polynomial is

$$p(r) = r^2 - 2r + 1 = (r-1)^2$$

Thus $r = 1$ is a repeated root and

$$y_h = c_1 e^x + c_2 x e^x$$

We present two approaches find a particular solution.

Complex Variable Approach: Consider a companion equation

$$v'' - 2v' + v = 4\cos x$$

Let $w = v + iy$, multiply the original equation by i and add it to the equation for v to get

$$w'' - 2w' + w = 4(\cos x + i\sin x) = 4e^{ix}$$

Since $p(i) = -1 - 2i + 1 = -2i \neq 0$, then

$$w_p = \frac{4e^{ix}}{-2i} = 2i(\cos x + i\sin x)$$

$y_p = \operatorname{Im} w_p = 2\cos x$. Thus

$$y_g = y_h + y_p = c_1 e^x + c_2 x e^x + 2\cos x$$

Without complex variables: Let

$$y_p = A\sin x + B\cos x$$

Then

$$y'_p = A\cos x - B\sin x$$
$$y''_p = -A\sin x - B\cos x$$

The DE implies

$$-A\sin x - B\cos x - 2A\cos x + 2B\sin x + A\sin x + B\cos x = 4\sin x$$

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Therefore

$$A = 0 \text{ and } B = 2$$

so that

$$y_p = A\sin x + B\cos x$$

as before.

$y'' - 2y' + y = 4\sin x$, Exact solution is: $\{2\cos x + C_5 e^x + C_6 x e^x\}$

2(b) (13 pts.) Find a general solution of

$$x^2 y'' - xy' = x^3 e^x, \quad x > 0$$

Solution: This is an Euler equation with $p = -1$ and $q = 0$. The indicial equation is

$$m^2 + (p-1)m + q = m^2 - 2m = 0$$

so $m = 0, 2$. The two linearly independent solutions are $x^0 = 1$ and x^2. Thus

$$y_h = c_1 x^0 + c_2 x^2 = c_1 + c_2 x^2$$

Let

$$y_p = v_1 + v_2 x^2$$

Then

$$v_1' y_1 + v_2' y_2 = v_1' + v_2' x^2 = 0$$

$$v_1' y_1' + v_2' y_2' = 2v_2' x = \frac{f}{a} = \frac{x^3 e^x}{x^2} = x e^x$$

$$W\left[1, x^2\right] = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x$$

Therefore

$$v_1' = \frac{\begin{vmatrix} 0 & x^2 \\ xe^x & 2x \end{vmatrix}}{2x} = -\frac{1}{2} x^2 e^x$$

so

$$v_1 = -\frac{1}{2}\int x^2 e^x dx = -\frac{1}{2} e^x \left(x^2 - 2x + 2\right)$$

$$v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & xe^x \end{vmatrix}}{2x} = \frac{1}{2} e^x$$

so

$$v_2 = \frac{1}{2} e^x$$

and

$$y_p = v_1 + v_2 x^2 = -\frac{1}{2} e^x \left(x^2 - 2x + 2\right) + \frac{1}{2} x^2 e^x = x e^x - e^x$$

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$$y_g = y_h + y_p = c_1 + c_2x^2 + xe^x - e^x$$

$x^2y'' - xy' = x^3e^x$, Exact solution is: $\left\{C_9x^2 - e^x - \frac{1}{2}C_8 + xe^x\right\}$

3. (a) (10 pts.) Let

$$g(t) = \begin{cases} e^t & \text{for } 0 \le t \le 2 \\ \\ 3 & \text{for } 2 < t < \infty \end{cases}$$

Use the definition of the Laplace transform to find $\mathcal{L}\{g(t)\}$.

Solution: Fro $s > 0$

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \int_0^\infty e^{-st}g(t)dt = \int_0^2 e^{-st}e^t dt + \int_2^\infty 3e^{-st}dt \\
&= \int_0^2 e^{(1-s)t}dt + 3\lim_{R\to\infty}\int_2^R e^{-st}dt \\
&= \frac{e^{(1-s)t}}{1-s}\bigg|_0^2 - \frac{3}{s}\lim_{R\to\infty} e^{-st}\big|_2^R \\
&= \frac{e^{2(1-s)}}{1-s} - \frac{1}{1-s} - \frac{3}{s}\lim_{R\to\infty}\left[e^{-sR} - e^{2s}\right] \\
&= \frac{e^{2(1-s)}}{1-s} - \frac{1}{1-s} + \frac{3}{s}e^{-2s}
\end{aligned}$$

(b) (15 pts.) Solve using Laplace Transforms:

$$y'' - 3y' + 4y = 0 \qquad y(0) = 1,\ y'(0) = 5$$

Solution: Taking the Laplace transform of both sides of the DE yields

$$\mathcal{L}\left\{y''\right\} - 3\mathcal{L}\left\{y'\right\} + 4\mathcal{L}\{y\} = \mathcal{L}\{0\} = 0$$

so that

$$s^2Y(s) - s(1) - 5 - 3[sY(s) - 1] + 4Y(s) = 0$$

Solving for $Y(s)$, we get

$$Y(s) = \frac{s+2}{s^2 - 3s + 4}$$

$$\frac{s+2}{s^2-3s+4} = \frac{s+2}{\left(s-\frac{3}{2}\right)^2 + \frac{7}{4}} = \frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} + \frac{\frac{7}{2}}{\left(s-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$$

$$= \frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} + \sqrt{7}\,\frac{\frac{\sqrt{7}}{2}}{\left(s-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$$

Hence

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$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+2}{s^2-3s+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\} + \sqrt{7}\,\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{7}}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}$$

$$= e^{\frac{3}{2}t}\cos\frac{\sqrt{7}}{2}t + \sqrt{7}\,e^{\frac{3}{2}t}\sin\frac{\sqrt{7}}{2}t$$

4.) a.) (10 pts.) Use separation of variables, $u(x,t) = X(x)T(t)$, to find two ordinary differential equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$5x^2t^7\frac{\partial^3 u}{\partial x^2\partial t} + (x+2)^3(t^2+10)^5\frac{\partial^2 u}{\partial t^2} = 0.$$

Note: Do **not** solve these ordinary differential equations.
Solution:

$$u_{xxt} = X''T' \quad u_{tt} = XT''$$

The DE becomes

$$5x^2t^7X''T' = -(x+2)^3(t^2+10)^5XT''$$

$$\frac{5x^2X''}{(x+2)^3X} = -\frac{(t^2+10)^5T''}{t^7T'} = k, \text{ where } k \text{ is a constant.}$$

Thus

$$5x^2X'' - k(x+2)^3X = 0$$
$$(t^2+10)^5T'' + kt^7T' = 0$$

b.) (15 pts.) Find the eigenvalues and eigenfunctions for

$$y'' - 4\lambda y' + 4\lambda^2 y = 0 \quad y'(1) = 0, \quad y(2) + 2y'(2) = 0$$

Solution: The characteristic polynomial is

$$p(r) = r^2 - 4\lambda r + 4\lambda^2 = (r-2\lambda)^2$$

Thus $r = 2\lambda$ is a repeated root and

$$y = c_1e^{2\lambda x} + c_2xe^{2\lambda x}$$

Hence

$$y' = 2c_1\lambda e^{2\lambda x} + 2c_2\lambda xe^{2\lambda x} + c_2e^{2\lambda x}$$

$$y'(1) = 0 \quad \Rightarrow 2\lambda c_1 + (1+2\lambda)c_2 = 0$$

$$y(2) + 2y'(2) = 0 \ \Rightarrow c_1 + 2c_2 + 2(2c_1\lambda + 4c_2\lambda + c_2) = 0$$

or

$$(1+4\lambda)c_1 + (4+8\lambda)c_2 = 0$$

So we have the following two equations

$$2\lambda c_1 + (1+2\lambda)c_2 = 0$$
$$(1+4\lambda)c_1 + (4+8\lambda)c_2 = 0$$

These two equations will have a nontrivial solution if and only if

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$$\begin{vmatrix} 2\lambda & 1+2\lambda \\ 1+4\lambda & 4+8\lambda \end{vmatrix} = 8\lambda + 16\lambda^2 - \left(1 + 6\lambda + 8\lambda^2\right)$$

$$= 8\lambda^2 + 2\lambda - 1 = (4\lambda - 1)(2\lambda + 1)$$

Thus, for a nontrivial solution we want $\lambda = \frac{1}{4}$ or $-\frac{1}{2}$. These are the eigenvalues. When $\lambda = -\frac{1}{2}$ the equations for c_1 and c_2 become

$$-c_1 + 0c_2 = 0$$
$$-2c_1 + 0c_2 = 0$$

Thus $c_1 = 0$ and c_2 can be any constant. Hence we have the corresponding eigenfunction

$$y = c_2 x e^{-x}$$

For $\lambda = \frac{1}{4}$ the equations for c_1 and c_2 become

$$\frac{1}{2}c_1 + \frac{3}{2}c_2 = 0$$
$$2c_1 + 6c_2 = 0$$

Thus $c_1 = -3c_2$ and the corresponding eigenfunction is

$$y = c_2(-3 + x)e^{\frac{1}{2}x}$$

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5. (a) (15 pts.) Find the first four nonzero terms of the Fourier *sine* series for the function

$$f(x) = \begin{cases} x & ; \quad 0 < x < \pi/2 \\ 0 & ; \quad \pi/2 < x < \pi \end{cases}$$

Be sure to give the Fourier series with these terms in it.

Solution: The Fourier sine series for $f(x)$ can be represented by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin(nx)dx\right]$$

We integrate by parts with $u = x$, $dv = \sin(nx)dx$ and $v = -\frac{1}{n}\cos(nx)$ to obtain

$$b_n = \frac{2}{\pi}\left[-\frac{1}{n}x\cos(nx) + \int_0^{\pi/2}\frac{1}{n}\cos(nx)dx\right]$$

$$\Rightarrow b_n = \frac{2}{\pi}\left[-\frac{1}{n}x\cos(nx)]_0^{\pi/2} + \frac{1}{n^2}\sin(nx)]_0^{\pi/2}\right]$$

$$\Rightarrow b_n = \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos(\frac{n\pi}{2}) + \frac{1}{n^2}\sin(\frac{n\pi}{2})\right]$$

If $n = 1$, $\quad b_1 = \frac{2}{\pi}[1] = \frac{2}{\pi}$
If $n = 2$, $\quad b_2 = \frac{2}{\pi}[\frac{\pi}{4}] = \frac{1}{2}$
If $n = 3$, $\quad b_3 = \frac{2}{\pi}[-\frac{1}{9}] = -\frac{2}{9\pi}$
If $n = 4$, $\quad b_4 = \frac{2}{\pi}[-\frac{\pi}{8}] = -\frac{1}{4}$

By substituting into the series we obtain:

$$f(x) \sim \sum_{n=1}^{\infty} b_n\sin(nx) = \frac{2}{\pi}\sin x + \frac{1}{2}\sin(2x) - \frac{2}{9\pi}\sin(3x) - \frac{1}{4}\sin(4x) + \ldots$$

(b) (10 pts.) Sketch the graph of the function represented by the Fourier sine series in 5 (a) on $-\pi < x < 3\pi$.

$$f(x) = \begin{cases} x & ; \quad 0 < x < \pi/2 \\ 0 & ; \quad \pi/2 < x < \pi \end{cases}$$

$\frac{\pi}{2} = 1.5708$
$\pi = 3.1416$
$\frac{3\pi}{2} = 4.7124$
$2\pi = 6.2832$
$\frac{5\pi}{2} = 7.8540$

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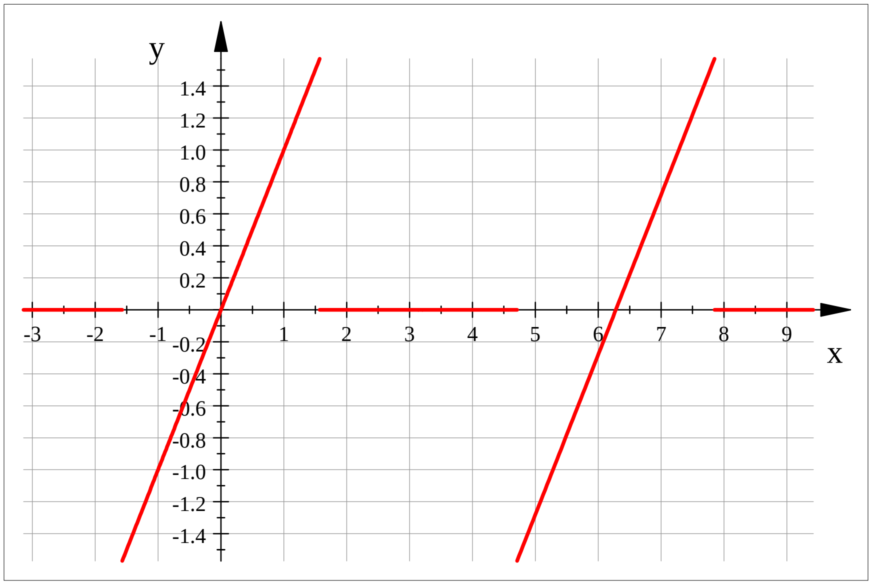
$3\pi = 9.4248$

x



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6 (25 pts.)

$$\text{PDE} \qquad u_t = 3u_{xx}$$

$$\text{BCs} \quad u(0,t) = 0 \qquad u_x(2,t) = 0$$

$$\text{IC} \quad u(x,0) = 2\sin(\frac{3\pi}{4}x) - 7\sin(\frac{5\pi}{4}x)$$

You must derive the solution. Your solution should not have any arbitrary constants in it. Show **all** steps.

Solution:

Assume

$$u(x,t) = X(x)T(t)$$

Then, $u_t = XT'$ and $u_{xx} = X''T$. By substituting into the PDE we obtain:

$$XT' = 3X''T$$

By separating the variables,

$$\frac{T'}{3T} = \frac{X''}{X} = \lambda$$

We obtain two ODEs:

$$X'' - \lambda X = 0$$

$$T' - 3\lambda T = 0$$

From the B.C.:

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u_x(2,t) = X'(2)T(t) = 0 \Rightarrow X'(2) = 0$$

We now solve the eigenvalue problem:

$$X'' - \lambda X = 0; \quad X(0) = 0, \quad X'(2) = 0$$

The auxiliary equation $r^2 - \lambda = 0$ implies that

$$r = \pm\sqrt{\lambda}$$

There are three cases: $\lambda = 0$, $\lambda > 0$, $\lambda < 0$

Case 1: $\lambda = 0$

$\Rightarrow r = 0$ is a double root.

$\Rightarrow X(x) = c_1 + c_2 x$

$0 = X(0) = c_1$

$X' = c_2$

$0 = X'(2) = c_2$

$\Rightarrow X \equiv 0$ (trivial)

Case 2: $\lambda > 0$, let $\lambda = k^2$ where $k \neq 0$

$\Rightarrow r = \pm k$

$\Rightarrow X(x) = c_1 e^{kx} + c_2 e^{-kx}$

$0 = X(0) = c_1 + c_2 \Rightarrow c_2 = -c_1$

$X' = kc_1 e^{kx} - kc_2 e^{-kx}$
$0 = X'(2) = kc_1 e^{2k} + kc_1 e^{-2k}$
$\Rightarrow 0 = kc_1 \left[e^{2k} + e^{-2k} \right]$
$\Rightarrow c_1 = c_2 = 0$ (trivial)

Case 3: $\lambda < 0$, let $\lambda = -k^2$ where $k \neq 0$
$\Rightarrow r = \pm ki$
$\Rightarrow X(x) = c_1 \cos kx + c_2 \sin kx$
$0 = X(0) = c_1$
$X' = -kc_1 \sin kx + kc_2 \cos kx$
$0 = X'(2) = kc_2 \cos 2k$
$\cos 2k = 0 \Rightarrow 2k = \frac{(2n+1)\pi}{2} \Rightarrow k = \frac{(2n+1)\pi}{4} \quad n = 0, 1, 2, \ldots$

We obtain

$$X_n = c_n \sin\left(\frac{(2n+1)\pi}{4} x \right)$$

With $\lambda_n = -\frac{(2n+1)^2\pi^2}{16}$, the second ODE becomes

$$T' + \frac{3(2n+1)^2\pi^2}{16} T = 0$$

This is a first-order separable equation. By separating the variables we obtain

$$\frac{dT}{T} = -\frac{3(2n+1)^2\pi^2}{16} dt$$

By integrating,

$$\ln T = -\frac{3(2n+1)^2\pi^2}{16} t$$

or

$$T_n = e^{-\frac{3(2n+1)^2\pi^2}{16} t}$$

The general solution is:

$$u(x,t) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{(2n+1)\pi}{4} x \right) e^{-\frac{3(2n+1)^2\pi^2}{16} t}$$

From the initial condition:

$$u(x,0) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{(2n+1)\pi}{4} x \right) = 2\sin(\frac{3\pi}{4} x) - 7\sin(\frac{5\pi}{4} x)$$

If $2n + 1 = 3 \Rightarrow n = 1 \Rightarrow c_1 = 2$
If $2n + 1 = 5 \Rightarrow n = 2 \Rightarrow c_2 = -7$
The other c_n's are 0.
Substituting into the general solution yields:

$$u(x,t) = 2\sin(\frac{3\pi}{4} x) e^{-\frac{27\pi^2}{16} t} - 7\sin(\frac{5\pi}{4} x) e^{-\frac{75\pi^2}{16} t}$$

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7. (a) (10 pts.) Solve

$$y'' - 4y' + 13y = 0; \quad y(0) = 0, \quad y(\pi) = 0$$

Solution:

The auxiliary equation is:

$$p(r) = r^2 - 4r + 13 = 0$$

This implies that

$$r = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

The general solution is:

$$y = c_1 e^{2x} \cos 3x + c_2 e^{2x} \sin 3x$$

By substituting the boundary conditions:

$$0 = y(0) = c_1$$

$$0 = y(\pi) = -c_1 e^{2\pi} \Rightarrow c_1 = 0$$

This implies that c_2 is an arbitrary constant. The solution satisfying the boundary conditions is:

$$y = Ce^{2x} \sin 3x; \qquad C \text{ arbitrary}$$

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(b) (15 pts.) Find the first 6 nonzero terms of the power series solution about $x = 0$ for the DE:

$$y'' + 2x^2 y' + 2xy = 0$$

Be sure to give the recurrence relation.

Solution:

Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

By substituting into the DE:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^{n+1} + \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

Let $n = k+3$ in the first sum and $n = k$ in the second and third sums to obtain

$$\sum_{k=-1}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} + \sum_{k=1}^{\infty} 2k a_k x^{k+1} + \sum_{k=0}^{\infty} 2 a_k x^{k+1} = 0$$

or

$$2a_2 + 6a_3 x + \sum_{k=1}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} + \sum_{k=1}^{\infty} 2k a_k x^{k+1} + 2a_0 x + \sum_{k=1}^{\infty} 2 a_k x^{k+1} = 0$$

By equating coefficients:

$$2a_2 = 0 \Rightarrow a_2 = 0$$

$$6a_3 + 2a_0 = 0 \Rightarrow a_3 = -\frac{1}{3} a_0$$

The recurrence relation is:

$$(k+3)(k+2) a_{k+3} + 2k a_k + 2a_k = 0$$

or

$$a_{k+3} = -\frac{2(k+1)}{(k+3)(k+2)} a_k; \qquad k = 1, 2, 3, \ldots$$

From the recurrence relation:

$$a_4 = -\frac{2(2)}{(4)(3)} a_1 = -\frac{1}{3} a_1$$

$$a_5 = -\frac{2(3)}{(5)(4)} a_2 = 0$$

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$$a_6 = -\frac{2(4)}{(6)(5)}a_3 = -\frac{4}{15}a_3 = \frac{4}{45}a_0$$

$$a_7 = -\frac{2(5)}{(7)(6)}a_4 = -\frac{5}{21}a_4 = \frac{5}{63}a_1$$

By substituting into the power series we obtain:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x - \frac{1}{3}a_0 x^3 - \frac{1}{3}a_1 x^4 + \frac{4}{45}a_0 x^6 + \frac{5}{63}a_1 x^7 + \ldots$$

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8 (a) (12 pts.) Solve

$$x\frac{dy}{dx} = 3y + x^2y^{-3}$$

Solution:

This is a Bernoulli equation. We first rewrite it in standard form:

$$\frac{dy}{dx} - \frac{3}{x}y = xy^{-3}$$

Multiply both sides by y^3

$$y^3\frac{dy}{dx} - \frac{3}{x}y^4 = x$$

Let $z = y^4$

$\Rightarrow \frac{dz}{dx} = 4y^3\frac{dy}{dx}$

By substituting into the DE,

$$\frac{1}{4}\frac{dz}{dx} - \frac{3}{x}z = x$$

or

$$\frac{dz}{dx} - \frac{12}{x}z = 4x$$

This is a first-order linear equation. The integrating factor is:

$$\mu(x) = e^{\int -\frac{12}{x}dx} = e^{-12\ln x} = e^{\ln(x^{-12})} = x^{-12}$$

By multiplying both sides by $\mu(x)$, we obtain:

$$x^{-12}\frac{dz}{dx} - 12x^{-13}z = 4x^{-11}$$

$$\Rightarrow \frac{d}{dx}(x^{-12}z) = 4x^{-11}$$

$$\Rightarrow x^{-12}z = -\frac{2}{5}x^{-10} + C$$

$$\Rightarrow z = -\frac{2}{5}x^2 + Cx^{12}$$

$$\Rightarrow y^4 = -\frac{2}{5}x^2 + Cx^{12}$$

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(b) (13 pts.) Find

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+6}{s(s-1)(s-3)}\right\}$$

Solution:

$$\hat{f}(s) = \frac{s^2+s+6}{s(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-3}$$

By multiplying both sides by $s(s-1)(s-3)$, we obtain:

$$s^2+s+6 = A(s-1)(s-3) + Bs(s-3) + Cs(s-1)$$

If $s = 0$,

$$6 = 3A \Rightarrow A = 2$$

If $s = 1$,

$$8 = -2B \Rightarrow B = -4$$

If $s = 3$,

$$18 = 6C \Rightarrow C = 3$$

This implies that

$$\hat{f}(s) = \frac{2}{s} - \frac{4}{s-1} + \frac{3}{s-3}$$

or

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+6}{s(s-1)(s-3)}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = 2 - 4e^t + 3e^{3t}$$

Name___________________________ Lecturer ________________________

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$		$s > a$
$\sin bt$	$\dfrac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int x \cos bx dx = \frac{1}{b^2}(\cos bx + bx \sin bx) + C$
$\int x \sin bx = \frac{1}{b^2}(\sin bx - bx \cos bx) + C$
$\int \left(\frac{e^{-t}}{1+e^t}\right) dt = -e^{-t} - \ln(e^t) + \ln(1+e^t) + C$
$\int xe^{ax} dx = \frac{1}{a^2}(axe^{ax} - e^{ax}) + C$
$\int x^2 e^{ax} dx = \frac{1}{a^3}e^{ax}\left(a^2x^2 - 2ax + 2\right) + C$
$\int u dv = uv - \int v du$