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Ma 221 **Final Exam Solutions** **5/18/10**

Print Name: _______________________________

Lecture Section: _________

I pledge my honor that I have abided by the Stevens Honor System.

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This exam consists of 8 problems. You are to solve all of these problems. The point value of each problem is indicated. The total number of points is 200.

If you need more work space, continue the problem you are doing on the **other side of the page it is on**. Be sure that you do all problems.

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

There are tables giving Laplace transforms and integrals at the end of the exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

Total Score ________

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1. Solve
(a) (8 pts)

$$x^2\left(1+y^2\right)dx + 2ydy = 0 \quad y(0) = 1$$

Solution: This equation is separable.

$$x^2dx + \frac{2y}{1+y^2}dy = 0$$

Integrating we have

$$\frac{x^3}{3} + \ln\left(1+y^2\right) = k$$

The initial condition implies $\ln 2 = k$ so

$$\frac{x^3}{3} + \ln\left(1+y^2\right) = \ln 2$$

(b) (7 pts) Solve

$$\left(3x^2y^2 + x^2\right)dx + \left(2x^3y + y^2\right)dy = 0$$

Solution: Since

$$M_y = 6x^2y = N_x$$

this equation is exact. Hence there exists $f(x,y)$ such that

$$f_x = M = 3x^2y^2 + x^2$$

Thus

$$f = x^3y^2 + \frac{1}{3}x^3 + g(y)$$

Then

$$f_y = 2x^3y + g'(y) = N = 2x^3y + y^2$$

Hence $g'(y) = y^2$ and $g(y) = \frac{1}{3}y^3$ so

$$f = x^3y^2 + \frac{1}{3}x^3 + \frac{1}{3}y^3$$

and the solution is given by

$$x^3y^2 + \frac{1}{3}x^3 + \frac{1}{3}y^3 = k$$

1 (c) (10 pts) Find a general solution of

$$y'' + 4y = 5e^t - 4t$$

Solution: The characteristic equation for the homogeneous equation is $p(r) = r^2 + 4 = 0,$ so $r = \pm 2i$. Thus

$$y_h = c_1 \sin 2t + c_2 \cos 2t$$

To find y_{p_1} for $5e^t$ we note that $p(1) = 5 \neq 0$ so

$$y_{p_1} = \frac{5e^t}{5} = e^t$$

To find y_{p_2} for $-4t$ we let

$$y_{p_2} = At^2 + Bt + C$$

Differentiating and substituting into $y'' + 4y = -4t$ yields

$$2A + 4At^2 + 4Bt + 4C = -4t$$

so $B = -1.$ and $A = C = 0.$ Thus $y_{p_2} = -t.$ A general solution is

$$y = y_h + y_{p_1} + y_{p_2} = c_1 \sin 2t + c_2 \cos 2t + e^t - t$$

2. (a) (12 pts) Find a general solution of

$$y'' - 4y' + 3y = 6 + 20\cos t$$

Solution: $p(r) = r^2 - 4r + 3 = (r-3)(r-1)$ so $r = 3, 1.$ Thus

$$y_h = c_1 e^{3t} + c_2 e^t$$

To find a particular solution for $20\cos t$ we present two methods.

I. Using complex variables

Consider the equation

$$y'' - 4y' + 3y = 20\cos t$$

and the auxiliary equation

$$v'' - 4v' + 3v = 20\sin t$$

Multiply the second equation by i and add the two equations and let $w = y + iv$ to get

$$w'' - 4w' + 3w = 20(\cos t + i\sin t) = 20e^{it}$$

Since $p(i) = i^2 - 4i + 3 = 2 - 4i \neq 0,$ then

$$w_p = \frac{20e^{it}}{2-4i} = \frac{10e^{it}}{1-2i}$$

We want the real part of w_p.

$$w_p = \frac{10e^{it}}{1-2i} \times \frac{1+2i}{1+2i} = 2e^{it}(1+2i) = 2(\cos t + i\sin t)(1+2i)$$

Hence y_{p_1} which is the real part of w_p is

$$y_{p_1} = 2\cos t - 4\sin t$$

y_{p_2} for 6 is 2. Thus

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^{3t} + c_2 e^t + 2\cos t - 4\sin t + 2$$

SNB check: $y'' - 4y' + 3y = 6 + 20\cos t$, Exact solution is: $\left\{2\cos t - 4\sin t + C_3 e^{3t} + C_2 e^t + 2\right\}$

II. Without complex variables

$$y_h = A\cos t + B\sin t$$
$$y_h' = -A\sin t + B\cos t$$
$$y_h'' = -A\cos t - B\sin t$$

The DE implies

$$-A\cos t - B\sin t + 4A\sin t - 4B\cos t + 3A\cos t + 3B\sin t = 20\cos t$$

Thus

$$-A - 4B + 3A = 2A - 4B = 20$$
$$-B + 4A + 3B = 4A + 2B = 0$$

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Multiply the second equation by 2 and add it to the first to get $10A = 20$ so $A = 2.$ From this we get that $B = -4$ and again

$$y_{p_1} = 2\cos t - 4\sin t$$

2(b) (13 pts.) Find a general solution of

$$y'' - 3y' + 2y = \frac{1}{1+e^{-x}}$$

Solution: We use variation of parameters. The characteristic equation is $p(r) = r^2 - 3r + 2 = (r-2)(r-1),$ so $r = 2, 1$ and

$$y_h = c_1 e^x + c_2 e^{2x}$$

Thus

$$y_p = v_1 e^x + v_2 e^{2x}$$

and the two equations for v_1' and v_2' are

$$v_1' e^x + v_2' e^{2x} = 0$$
$$v_1' e^x + 2v_2' e^{2x} = \frac{1}{1+e^{-x}}$$

$W\left[e^x, e^{2x}\right] = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}.$ Thus

$$v_1' = \frac{\begin{vmatrix} 0 & e^{2x} \\ \frac{1}{1+e^{-x}} & 2e^{2x} \end{vmatrix}}{W\left[e^x, e^{2x}\right]} = \frac{-\frac{e^{2x}}{1+e^{-x}}}{e^{3x}} = -\frac{e^{-x}}{1+e^{-x}}$$

Thus

$$v_1 = \ln(1+e^{-x})$$

$$v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{1+e^{-x}} \end{vmatrix}}{W\left[e^x, e^{2x}\right]} = \frac{\frac{e^x}{1+e^{-x}}}{e^{3x}} = \frac{e^{-2x}}{1+e^{-x}}$$

so

$$v_2 = \ln(1+e^{-x}) - e^{-x}$$

from the table.

$$\begin{aligned} y_p &= v_1 e^x + v_2 e^{2x} = e^x \ln(1+e^x) + e^{2x}(\ln(1+e^{-x}) - e^{-x}) \\ &= \left(e^x + e^{2x}\right)\ln(1+e^x) + e^x \end{aligned}$$

We may ignore the e^x in this particular solution, since e^x is a homogeneous solution. Thus we have

$$y = y_h + y_p = c_1 e^x + c_2 e^{2x} + \left(e^x + e^{2x}\right)\ln(1+e^x)$$

3. (a) (10 pts.) Let

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t \le 3 \\ e^{4t} & \text{for } 3 < t < \infty \end{cases}$$

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Use the definition of the Laplace transform to find $\mathcal{L}\{g(t)\}$ for $s > 4$.

Solution:

$$\begin{aligned}\mathcal{L}\{g(t)\} &= \int_0^\infty e^{-st}g(t)dt = \int_0^3 0 \cdot e^{-st}dt + \int_3^\infty e^{(4-s)t}dt \\ &= \lim_{R\to\infty}\left(\frac{e^{(4-s)t}}{4-s}\Bigg|_3^R \right) = \frac{1}{4-s}\left[e^{(4-s)R} - e^{(4-s)3} \right] \\ &= \frac{e^{12-3s}}{s-4} \quad \text{for } s > 4\end{aligned}$$

(b) (15 pts.) Solve using Laplace Transforms:

$$y'' + 2y' + y = 3te^{-t} \qquad y(0) = 4,\ \ y'(0) = 2$$

Solution: Taking Laplace transforms we have

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{3te^{-t}\}$$

so that

$$s^2Y(s) - 4s - 2 + 2[sY(s) - 4] + Y(s) = \frac{3}{(s+1)^2}$$

Note: We get the transform of $3te^{-t}$ from the table below by noting that the transform of t is $\frac{1}{s^2}$ and the $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f\}(s-a)$. Here $a = -1$ and $f(t) = t$.

Then

$$\left(s^2 + 2s + 1\right)Y(s) = 4s + 10 + \frac{3}{(s+1)^2}$$

so

$$Y(s) = \frac{4s+10}{(s+1)^2} + \frac{3}{(s+1)^4}$$

$$\begin{aligned}Y(s) &= \frac{4s+4}{(s+1)^2} + \frac{6}{(s+1)^2} + \frac{3}{(s+1)^4} \\ &= \frac{4}{s+1} + \frac{6}{(s+1)^2} + \frac{3}{(s+1)^4}\end{aligned}$$

From the table using the shift property we have

$$y(t) = 4e^{-t} + 6te^{-t} + \frac{1}{2}t^3e^{-t}$$

4.) a.) (10 pts.) Use separation of variables, $u(x,t) = X(x)T(t)$, to find two ordinary differential equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$2(x+1)^2t^3\frac{\partial^2 u}{\partial x \partial t} - (x+6)^4(t^4+10)^6\frac{\partial^2 u}{\partial x^2} = 0$$

Note: Do **not** solve these ordinary differential equations.

Solution: We have $u_x = X'T, u_{xt} = X'T', u_{xx} = X''T.$ Thus the PDE implies

$$2(x+1)^2t^3X'T' - (x+6)^4(t^4+10)^6X''T = 0$$

or

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$$\frac{2(x+1)^2}{(x+6)^4}\frac{X'}{X''} = \frac{(t^4+10)^6}{t^3}\frac{T}{T'} = k$$

Thus the two ODEs are

$$2(x+1)^2X' - k(x+6)^4X'' = 0$$
$$\left(t^4+10\right)^6T - kt^3T = 0$$

b.) (15 pts.) Find the eigenvalues and eigenfunctions for Find the eigenvalues and eigenfunctions for

$$x^2y'' + xy' + \lambda y = 0 \quad y(1) = y(e^{\pi}) = 0$$

Solution: The equation is an Euler equation. There are three cases to deal with, $\lambda > 0, \lambda = 0, \lambda < 0.$

I. $\lambda < 0.$ Let $\lambda = -\alpha^2,$ where $\alpha \neq 0.$ The indicial equation is $m^2 + (p-1)m + q = 0.$ Here $p = 1$ and $q = \lambda.$ Hence

$$m^2 - \alpha^2 = 0$$

so $m = \pm\alpha.$ Therefore

$$y = c_1x^{\alpha} + c_2x^{-\alpha}$$

The initial conditions imply

$$c_1 + c_2 = 0$$
$$c_1e^{\alpha\pi} + c_2e^{-\alpha\pi} = 0$$

From the first equation we see that $c_1 = -c_2$ so the second equation becomes

$$c_1(e^{\alpha\pi} - e^{-\alpha\pi}) = 0$$

Since $e^{\alpha\pi} - e^{-\alpha\pi} \neq 0$ for $\alpha \neq 0,$ then $c_1 = c_2 = 0$ and $y = 0.$

II. $\lambda = 0.$ The indicial equation $m^2 = 0$ and

$$y = c_1 + c_2\ln x$$

The condition $y(1) = 0$ implies that $c_1 = 0.$ The condition $y(e^{\pi}) = 0$ implies that $c_2 = 0,$ so again $y = 0.$

III.$\lambda > 0.$ Let $\lambda = \beta^2,$ where $\beta \neq 0.$ Then $m = \pm\beta i$ and

$$y = x^0[c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)]$$

The condition $y(1) = 0$ implies that $c_1\cos(\beta\ln 1) = 0,$ so $c_1 = 0.$ The condition

$$y(e^{\pi}) = c_2\sin\beta\pi = 0$$

Thus $\beta = n,$ where $n = 1,2,\ldots$ and $\lambda = n^2 \;\; n = 1,2,\ldots$ are the eigenvalues and

$$y_n(x) = a_n\sin(n\ln x) \;\; n = 1,2,\ldots$$

are the eigenfunctions.

5. (a) (15 pts.) Find the first four nonzero terms of the Fourier *cosine* series for the function

$$f(x) = x \;\text{ on } 0 < x < 1$$

Solution:

$$f(x) = \beta_0 + \sum_1^{\infty}\beta_n\cos\frac{n\pi x}{L}$$

where

$$\beta_0 = \frac{1}{L}\int_0^L f(x)dx \text{ and } \beta_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad n = 1,2,3,\ldots$$

Here $L = 1$ so

$$f(x) = \beta_0 + \sum_1^\infty \beta_n \cos n\pi x$$

$$\beta_0 = \frac{1}{1}\int_0^1 xdx = \frac{1}{2}$$

$$\beta_n = \frac{2}{1}\int_0^1 x\cos n\pi x dx = \frac{2}{n^2\pi^2}(\cos n\pi x + n\pi x\sin n\pi x)|_0^1$$

$$= \frac{2}{n^2\pi^2}(\cos n\pi - 1) = \frac{2}{n^2\pi^2}((-1)^n - 1) \quad n = 1,2,3,\ldots$$

Hence $\beta_1 = -\frac{4}{\pi^2}$, $\beta_2 = 0$, $\beta_3 = -\frac{4}{9\pi^2}$, $\beta_4 = 0$, $\beta_5 = -\frac{4}{25\pi^2}$

Therefore

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\cos\pi x - \frac{4}{9\pi^2}\cos 3\pi x - \frac{4}{25\pi^2}\cos 5\pi x$$

Note: The book gives the formulas

$$f(x) = \frac{\beta_0}{2} + \sum_1^\infty \beta_n\cos\frac{n\pi x}{L}$$

where

$$\beta_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad n = 0,1,2,3,\ldots$$

Using this formula we get

$$\beta_0 = \frac{2}{1}\int_0^1 xdx = 1$$

Therefore, the first term in the book's formula for the Fourier cosine series is $\frac{\beta_0}{2} = \frac{1}{2}$ as before.

(b) (10 pts.) Sketch the graph of the function represented by the Fourier cosine series in 5 (a) on $-3 < x < 3$.

x

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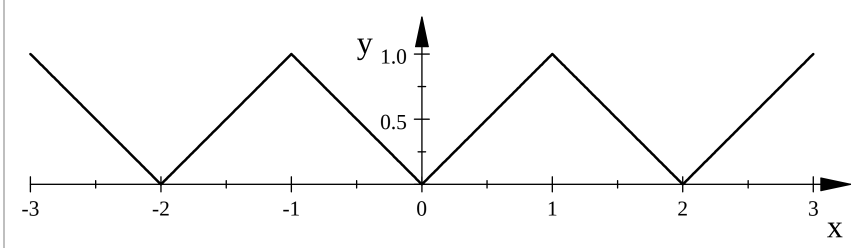

6 (25 pts.)

$$\text{PDE} \qquad u_{xx} - 16u_{tt} = 0$$

$$\text{BCs} \quad u(0,t) = 0 \qquad u_x(1,t) = 0$$

$$\text{IC} \quad u(x,0) = -6\sin\left(\frac{3\pi x}{2}\right) + 13\sin\left(\frac{11\pi x}{2}\right)$$

$$\text{IC} \quad u_t(x,0) = 0$$

You must derive the solution. Your solution should not have any arbitrary constants in it. Show **all** steps.

Solution: Let $u(x,t) = X(x)T(t)$. Then the PDE implies

$$X''T = 16XT''$$

or

$$\frac{X''}{X} = 16\frac{T''}{T} = -\lambda^2$$

since we will need sines and cosines in the X part of the solution.

Thus

$$X'' + \lambda^2 X = 0$$

$$T'' + \frac{\lambda^2}{16}T = 0$$

The BCs are

$$X(0) = X'(1) = 0$$

$$X(x) = a_n \sin\lambda x + b_n \cos\lambda x$$

$X(0) = 0$ implies that $b_n = 0,$ so

$$X(x) = a_n \sin\lambda x$$

$$X'(x) = a_n\lambda\cos\lambda x$$

so

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$$X'(1) = a_n \lambda \cos\lambda = 0$$

Hence $\lambda = \frac{2n+1}{2}\pi, \ \ n = 0, 1, 2, \ldots$ and

$$X_n(x) = A_n \sin\left(\frac{2n+1}{2}\right)\pi x \quad n = 0, 1, 2, \ldots$$

Also

$$T'' + \frac{\lambda^2}{16}T = T'' + \frac{(2n+1)^2\pi^2}{64}T = 0$$

$$T_n(t) = c_n \sin\left(\frac{2n+1}{8}\right)\pi t + d_n \cos\left(\frac{2n+1}{8}\right)\pi t$$

$u_t(x,0) = 0$ implies that $c_n = 0$ and

$$T_n(t) = d_n \cos\left(\frac{2n+1}{8}\right)\pi t$$

Thus

$$u_n(x.t) = B_n \sin\left(\frac{2n+1}{2}\right)\pi x \cos\left(\frac{2n+1}{8}\right)\pi t \quad n = 0, 1, 2, \ldots$$

Let

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x.t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2}\right)\pi x \cos\left(\frac{2n+1}{8}\right)\pi t$$

$$u(x,0) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2}\right)\pi x = -6\sin\left(\frac{3\pi x}{2}\right) + 13\sin\left(\frac{11\pi x}{2}\right)$$

Therefore $B_1 = -6, B_5 = 13$ and $B_n = 0$ for $n \neq 1, 5$ so

$$u(x,t) = -6\sin\left(\frac{3\pi x}{2}\right)\cos\left(\frac{3\pi}{8}\right)t + 13\sin\left(\frac{11\pi x}{2}\right)\cos\left(\frac{11\pi}{8}\right)t$$

7. (a) (10 pts.) Solve

$$y'' + 4y' + 20y = 0$$

Solution: The characteristic equation is $r^2 + 4r + 20 = 0$ so

$$r = \frac{-4 \pm \sqrt{4^2 - 4(1)(20)}}{2} = \frac{-4 \pm \sqrt{-64}}{2} = -2 \pm 4i$$

Thus

$$y(x) = c_1 e^{-2x}\cos 4t + c_2 e^{-2x}\sin 4t$$

(b) (15 pts.) Find the first 5 nonzero terms of the power series solution about $x = 0$ for the DE:

$$\left(4 - x^2\right)y' + y = 0$$

Be sure to give the recurrence relation.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

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$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

The DE implies

$$\left(4 - x^2\right) \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$4 \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Let $n - 1 = k$ in the first sum, that is $n = k + 1$ and let $j = n + 1$ in the second sum, that is $n = j - 1$
Then we have

$$4 \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{j=2}^{\infty} (j-1) a_{j-1} x^j + \sum_{n=0}^{\infty} a_n x^n = 0$$

Since k, j, n are "dummy" place keepers we may replace them by m to get

$$4 \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m - \sum_{m=2}^{\infty} (m-1) a_{m-1} x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$4a_1 + a_0 + (8a_2 + a_1)x + \sum_{m=2}^{\infty} [4(m+1)a_{m+1} - (m-1)a_{m-1} + a_m] x^m = 0$$

This implies that

$$a_1 = -\frac{1}{4} a_0$$
$$a_2 = -\frac{1}{8} a_1 = \frac{1}{32} a_0$$

and the recurrence relation

$$a_{m+1} = \frac{(m-1)a_{m-1} - a_m}{4(m+1)} \quad m = 2, 3, \ldots$$

Therefore letting $m = 2$

$$a_3 = \frac{a_1 - a_2}{4(3)} = \frac{-\frac{1}{4} - \frac{1}{32}}{12} a_0 = -\frac{3}{128} a_0$$

Letting $m = 3$

$$a_4 = \frac{2a_2 - a_3}{4(4)} = \frac{\frac{1}{16} + \frac{3}{128}}{16} a_0 = \frac{11}{2048} a_0$$

Thus

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \left[1 - \frac{1}{4} x + \frac{1}{32} x^2 - \frac{3}{128} x^3 + \frac{11}{2048} x^4 + \cdots \right]$$

8 (a) (10 pts.) Solve

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{xy^2} \quad x > 0$$

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Solution: We may rewrite the equation as

$$y^2y' + \frac{y^3}{x} = \frac{1}{x}$$

Let $z = y^3$ so $z' = 3y^2y'$ and the above DE becomes

$$\frac{1}{3}z' + \frac{z}{x} = \frac{1}{x}$$

or

$$z' + \frac{3}{x}z = \frac{3}{x}$$

The integrating factor is $e^{\int \frac{3}{x}dx} = x^3.$ Multiplying the DE by this we have

$$x^3z' + 3x^2z = 3x^2$$

or

$$\frac{d\left(x^3z\right)}{dx} = 3x^2$$

so

$$x^3z = x^3 + c$$

or

$$y^3 = z = 1 + \frac{c}{x^3}$$

so

$$y = \left(1 + \frac{c}{x^3}\right)^{\frac{1}{3}}$$

(b) (15 pts.) Find

$$\mathcal{L}^{-1}\left\{\frac{2s^3 - 4s - 8}{s(s-1)(s^2+4)}\right\}$$

Solution: I. Without complex variables

$$\frac{2s^3 - 4s - 8}{s(s-1)(s^2+4)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$s = 0$ implies $A = \frac{-8}{-4} = 2,\ s = 1$ implies $\frac{-10}{5} = -2 = B$

so

$$\frac{2s^3 - 4s - 8}{s(s-1)(s^2+4)} = \frac{2}{s} + \frac{-2}{s-1} + \frac{Cs+D}{s^2+4}$$

Setting $s = -1$ we have

$$\frac{-2+4-8}{(-1)(-2)(5)} = -2 + 1 + \frac{-C+D}{5}$$

or

$$\frac{-3}{5} = -1 + \frac{-C+D}{5}$$

so

$$2 = -C + D$$

Let $s = 2$

$$0 = 1 - 2 + \frac{2C+D}{8}$$

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or

$$8 = 2C + D$$

Thus $C = 2, D = 4$ and

$$\frac{2s^3 - 4s - 8}{s(s-1)(s^2+4)} = \frac{2}{s} + \frac{-2}{s-1} + \frac{2s+4}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{2s^3 - 4s - 8}{s(s-1)(s^2+4)}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= 2 - 2e^t + 2\cos 2t + 2\sin 2t$$

II. Using Complex variables

$$\frac{2s^3 - 4s - 8}{s(s-1)(s^2+4)} = \frac{2s^3 - 4s - 8}{s(s-1)(s+2i)(s-2i)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2i} + \frac{D}{s-2i}$$

As before setting $s = 0$ and $s = 1$ gives $A = 2, B = -2$

Let $s = 2i$

$$\frac{-16i - 8i - 8}{2i(2i-1)(4i)} = \frac{-24i - 8}{-8(2i-1)} = \frac{3i+1}{2i-1} = D$$

$$D = -\frac{3i+1}{1-2i} \times \frac{1+2i}{1+2i} = -\frac{(1+3i)(1+2i)}{5} = -\frac{-5+5i}{5} = 1 - i$$

Let $s = -2i$

$$\frac{16i + 8i - 8}{-2i(-2i-1)(-4i)} = \frac{24i - 8}{8(2i+1)} = \frac{3i-1}{2i+1}C$$

$$C = -\frac{1-3i}{1+2i} \times \frac{1-2i}{1-2i} = -\frac{-5-5i}{5} = 1 + i$$

$$\frac{2s^3 - 4s - 8}{s(s-1)(s^2+4)} = \frac{2}{s} + \frac{-2}{s-1} + (1+i)\left(\frac{1}{s+2i}\right) + (1-i)\left(\frac{1}{s-2i}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{2s^3 - 4s - 8}{s(s-1)(s^2+4)}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + (1+i)\mathcal{L}^{-1}\left\{\frac{1}{s+2i}\right\} + (1-i)\mathcal{L}^{-1}\left\{\frac{1}{s-2i}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s^3 - 4s - 8}{s(s-1)(s^2+4)}\right\} = 2 - 2e^t + (1+i)e^{-2it} + (1-i)e^{2it}$$

$$= 2 - 2e^t + (1+i)(\cos 2t - i\sin 2t) + (1-i)(\cos 2t + i\sin 2t)$$

$$= 2 - 2e^t + 2\cos 2t + 2\sin 2t$$

as before.

Name___________________________ Lecturer _______________________

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int x \cos bx dx = \frac{1}{b^2}(\cos bx + bx \sin bx) + C$
$\int x \sin bx dx = \frac{1}{b^2}(\sin bx - bx \cos bx) + C$
$\int \left(\frac{e^{-t}}{1+e^{-t}}\right) dt = -\ln(1+e^{-t}) + C$
$\int \left(\frac{e^{-2t}}{1+e^{-t}}\right) dt = \ln(1+e^{-t}) - e^{-t} + C$
$\int x e^{ax} dx = \frac{1}{a^2}(axe^{ax} - e^{ax}) + C$