

Ma 221

Final Exam Solutions

5/15/12

Print Name: _____

Lecture Section: _____

1. Solve

(a) (8 pts)

$$2xy^3 dx - (1 - y^2) dy = 0 \quad y(1) = 1$$

Solution: This equation is separable. We rearrange and integrate.

$$\begin{aligned} 2x dx &= \left(\frac{1}{y^3} - \frac{1}{y} \right) dy \\ x^2 &= \frac{-1}{2y^2} - \ln(|y|) + c \end{aligned}$$

We find the constant from the initial condition.

$$\begin{aligned} 1 &= \frac{-1}{2} + c \\ c &= \frac{3}{2} \end{aligned}$$

Combining these, the implicit solution is

$$x^2 + \frac{1}{2y^2} + \ln y = \frac{3}{2}.$$

(b) (7 pts) Solve

$$(8x^3y^3 + 4x^3) dx + (6x^4y^2 + 4y^3) dy = 0$$

Solution: We consider this d. e. to be in the form

$$M(x, y) dx + N(x, y) dy = 0.$$

Since

$$M_y = 24x^3y^2 = N_x$$

the d. e is exact. I.e , there is a function $F(x, y)$ such that

$$dF = F_x dx + F_y dy = M dx + N dy.$$

Since a solution to the d.e. is given by $F(x, y) = c$, we will find such a function, $F(x, y)$.

$$\begin{aligned} F_x &= M(x, y) = (8x^3y^3 + 4x^3) \\ F &= \int (8x^3y^3 + 4x^3) dx \\ &= 2x^4y^3 + x^4 + g(y) \\ F_y &= 6x^4y^2 + g'(y) = N(x, y) = (6x^4y^2 + 4y^3) \end{aligned}$$

Hence

$$\begin{aligned} g'(y) &= 4y^3 \\ g(y) &= y^4 \\ F(x, y) &= 2x^4y^3 + x^4 + y^4 \end{aligned}$$

and an implicit solution is

$$2x^4y^3 + x^4 + y^4 = c.$$

1 (c) (10 pts) Find a general solution of

$$y'' + 2y' - 3y = 8e^t + 18t$$

Solution: The characteristic equation is

$$p(r) = r^2 + 2r - 3 = (r + 3)(r - 1) = 0.$$

Thus the solution of the homogeneous equation is

$$y_h = c_1 e^{-3t} + c_2 e^t.$$

To find a particular solution, y_{p1} , for $8e^t$, we compute $p'(1) = 2 + 2 = 4$ and have

$$y_{p1} = \frac{8te^t}{4} = 2te^t.$$

To find y_{p2} for $18t$, we set

$$y_{p2} = At + B.$$

Substitution into the d.e gives

$$2A - 3(At + B) = -3At + (2A - 3B) = 18t.$$

So $A = -6$ and $B = -4$ and $y_{p2} = -6t - 4$. combining everything, we have that a general solution is given by

$$\begin{aligned} y &= y_h + y_{p1} + y_{p2} \\ &= c_1 e^{-3t} + c_2 e^t + 2te^t - 6t - 4. \end{aligned}$$

2. (a) (12 pts) Find a general solution of

$$y'' + 4y' + 5y = 16 \sin t$$

Solution: The characteristic equation is

$$r^2 + 4r + 5 = (r + 2)^2 + 1 = 0.$$

So

$$\begin{aligned} (r + 2)^2 &= -1 \\ r + 2 &= \pm i \\ r &= -2 \pm i \end{aligned}$$

and a general solution of the homogeneous equation is

$$y_h = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t.$$

To find a particular solution of the given equation using the method of undetermined coefficients, we proceed as follows.

$$\begin{aligned} y &= A \cos t + B \sin t \\ y' &= -A \sin t + B \cos t \\ y'' &= -A \cos t - B \sin t \end{aligned}$$

Substituting these into the d.e. gives

$$(-A + 4B + 5A)\cos t + (-B - 4A + 5B)\sin t = 16\sin t$$

$$4A + 4B = 0$$

$$-4A + 4B = 16$$

Add the two equations to obtain $8B = 16$, so $B = 2$. Then from the first, $A = -2$. Combining everything, we have

$$y_p = -2\cos t + 2\sin t$$

$$y = y_h + y_p$$

$$= (c_1 e^{-2t} - 2)\cos t + (c_2 e^{-2t} + 2)\sin t$$

SNB check: $y'' + 4y' + 5y = 16\sin t$, Exact solution is: $2\sin t - 2\cos t + C_1(\cos t)e^{-2t} + C_2(\sin t)e^{-2t}$

2(b) (13 pts.) Find a general solution of

$$y''(\theta) + 16y(\theta) = \tan 4\theta$$

Solution: The characteristic equation is $r^2 + 16 = 0$, so the roots, $r = \pm 4i$, are complex and the solution to the homogeneous equation is

$$y_h = c_1 \cos(4\theta) + c_2 \sin(4\theta).$$

To find a particular solution, we use the method of variation of parameters. We set

$$y = v_1 \cos(4\theta) + v_2 \sin(4\theta)$$

where v_1 and v_2 are variable. With the simplifying assumption that eliminates second derivatives, the equations for v_1 and v_2 are

$$v_1' y_1 + v_2' y_2 = 0 \tag{A}$$

$$v_1' y_1' + v_2' y_2' = \frac{f}{a}$$

where y_1 and y_2 are the solutions to the homogeneous equation and $\frac{f}{a}$ is the right hand side of the d.e. For our problem, these equations are

$$\cos(4\theta)v_1' + \sin(4\theta)v_2' = 0 \tag{B}$$

$$-4\sin(4\theta)v_1' + 4\cos(4\theta)v_2' = \tan(4\theta)$$

Multiply the first equation by $4\sin 4\theta$, the second equation by $\cos 4\theta$ and add to obtain

$$4(\sin^2 4\theta + \cos^2 4\theta)v_2' = 4v_2' = \sin 4\theta$$

$$v_2' = \frac{1}{4} \sin 4\theta$$

$$v_2 = \int \frac{1}{4} \sin 4\theta d\theta$$

$$= -\frac{1}{16} \cos 4\theta + c_2$$

Substituting v_2' into the first of the equations in (A), we obtain

$$\begin{aligned}
 \cos(4\theta)v_1' + \sin(4\theta)\left[\frac{1}{4}\sin 4\theta\right] &= 0 \\
 \cos(4\theta)v_1' &= -\left[\frac{1}{4}\sin^2 4\theta\right] \\
 v_1' &= \frac{-1}{4}\left[\frac{1 - \cos^2 4\theta}{\cos 4\theta}\right] \\
 &= \frac{-1}{4}(\sec 4\theta - \cos 4\theta) \\
 v_1 &= \frac{-1}{4}\int(\sec 4\theta - \cos 4\theta)d\theta \\
 &= \frac{-1}{16}\ln|\sec 4\theta + \tan 4\theta| + \frac{1}{16}\sin 4\theta + c_1
 \end{aligned}$$

The integral of the secant is obtained from the integral table. finally, we combine results to obtain the desired solution. The homogeneous solution is included via the constants of integration, so they have been labeled as before.

$$\begin{aligned}
 y &= v_1 \cos(4\theta) + v_2 \sin(4\theta) \\
 &= \left[\frac{-1}{16}\ln|\sec 4\theta + \tan 4\theta| + \frac{1}{16}\sin 4\theta + c_1\right]\cos 4\theta + \left[-\frac{1}{16}\cos 4\theta + c_2\right]\sin 4\theta \\
 &= \left[\frac{-1}{16}\ln|\sec 4\theta + \tan 4\theta|\right]\cos 4\theta + c_1 \cos 4\theta + c_2 \sin 4\theta
 \end{aligned}$$

3. (a) (10 pts.) Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} e^{2t} & 0 \leq t < 3 \\ 1 & 3 < t \end{cases}$$

for $s > 2$.

Solution:

$$\begin{aligned}
 \mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}f(t)dt = \int_0^3 e^{-st}e^{2t}dt + \int_3^\infty e^{-st}(1)dt \\
 &= \frac{e^{(2-s)t}}{2-s} \Big|_0^3 - \lim_{R \rightarrow \infty} \frac{e^{-st}}{s} \Big|_3^R = \frac{1 - e^{-3(2-s)}}{s-2} + \frac{e^{-3s}}{s}
 \end{aligned}$$

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(b) (15 pts.) Solve using Laplace Transforms:

$$y'' + 2y' + y = t^2 + 4t \quad y(0) = 0, \quad y'(0) = -1$$

Solution: Taking Laplace transforms we have

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{t^2 + 4t\}$$

so that

$$\begin{aligned}
 s^2 Y(s) + 1 + 2[sY(s)] + Y(s) &= \frac{2}{s^3} + \frac{4}{s^2} \\
 (s^2 + 2s + 1)Y(s) + 1 &= \frac{4s + 2}{s^3} \\
 (s^2 + 2s + 1)Y(s) &= \frac{4s + 2}{s^3} - 1 \\
 &= \frac{-s^3 + 4s + 2}{s^3} \\
 Y(s) &= \frac{-s^3 + 4s + 2}{s^3(s^2 + 2s + 1)}
 \end{aligned}$$

We use the partial fractions method to get back to entries in the Laplace Transform table.

$$\begin{aligned}
 \frac{-s^3 + 4s + 2}{s^3(s + 1)^2} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s + 1} + \frac{E}{(s + 1)^2} \\
 -s^3 + 4s + 2 &= As^2(s + 1)^2 + Bs(s + 1)^2 + C(s + 1)^2 + Ds^3(s + 1) + Es^3
 \end{aligned}$$

Setting s to 0 and -1 gives two of the coefficients.

$$\begin{aligned}
 s = 0 &\Rightarrow 2 = C \\
 s = -1 &\Rightarrow 1 - 4 + 2 = -E \quad E = 1
 \end{aligned}$$

The rest can be obtained either by choosing some more values for s and producing equations or by equating the coefficients of the various powers of s . Some are easier to pick out than others. We have already used the constant which resulted when we set s to 0. The coefficient of s is

$$\begin{aligned}
 4 &= B + 2C = B + 4 \\
 B &= 0
 \end{aligned}$$

The coefficient of s^2 is

$$\begin{aligned}
 0 &= A + 2B + C = A + 2 \\
 A &= -2
 \end{aligned}$$

The coefficient of s^4 is

$$\begin{aligned}
 0 &= A + D = -2 + D \\
 D &= 2
 \end{aligned}$$

So, we have

$$\begin{aligned}
 Y(s) &= \frac{-s^4 + 4s + 2}{s^3(s + 1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s + 1} + \frac{E}{(s + 1)^2} \\
 &= \frac{-2}{s} + \frac{0}{s^2} + \frac{2}{s^3} + \frac{2}{s + 1} + \frac{1}{(s + 1)^2} \\
 y(t) &= \mathcal{L}^{-1} \left\{ \frac{-2}{s} + \frac{2}{s^3} + \frac{2}{s + 1} + \frac{1}{(s + 1)^2} \right\} \\
 &= -2 + t^2 + 2e^{-t} + te^{-t}
 \end{aligned}$$

4.) a.) (10 pts.) Use separation of variables, $u(x,t) = X(x)T(t)$, to find two ordinary differential equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$(t^2 + 1)u_{xx} + x^2(t^2 + 1)u_x - (x^2 + 1)u_{tt} = 0$$

Note: Do **not** solve these ordinary differential equations.

Solution: We have

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ u_x &= X'(x)T(t) \\ u_{xx} &= X''(x)T(t) \\ u_{tt} &= X(x)T''(t) \end{aligned}$$

Substitute and then separate.

$$\begin{aligned} (t^2 + 1)X''T + x^2(t^2 + 1)X'T - (x^2 + 1)XT'' &= 0 \\ (t^2 + 1)X''T + x^2(t^2 + 1)X'T &= (x^2 + 1)XT'' \\ \frac{X'' + x^2X'}{(x^2 + 1)X} &= \frac{T''}{(t^2 + 1)T} = k \end{aligned}$$

The last step - equating both sides to a constant - is based on having a function only of x on the left side of the equality and a function of t on the right side. Now, we extract the two ordinary differential equations.

$$\begin{aligned} X'' + x^2X' &= k(x^2 + 1)X \\ T'' &= k(t^2 + 1)T \end{aligned}$$

These can be rewritten as follows, but it's really not necessary.

$$\begin{aligned} X'' + x^2X' - k(x^2 + 1)X &= 0 \\ T'' - k(t^2 + 1)T &= 0 \end{aligned}$$

b.) (15 pts.) Find all eigenvalues (λ) and the corresponding eigenfunctions for the boundary value problem

$$y'' - 2y + \lambda y = 0 \quad y'(0) = y'(\pi) = 0$$

Solution: The characteristic equation of the d. e. is

$$\begin{aligned} r^2 - 2 + \lambda &= 0 \\ r^2 &= 2 - \lambda \\ r &= \pm \sqrt{2 - \lambda} \end{aligned}$$

There are three cases to be considered depending on whether the quantity under the radical is positive, zero or negative. We deal with each case in turn.

Case I: $2 - \lambda > 0$. Let's write $\mu^2 = 2 - \lambda$. Hence $r = \pm\mu$ and the solution to the d.e. is

$$\begin{aligned} y &= c_1 e^{\mu x} + c_2 e^{-\mu x} \\ y' &= \mu(c_1 e^{\mu x} - c_2 e^{-\mu x}). \end{aligned}$$

From $y'(0) = 0$,

$$c_1 = c_2.$$

From $y'(\pi) = 0$,

$$\mu c_1(e^{\mu\pi} + e^{-\mu\pi}) = 0.$$

Hence $c_1 = c_2 = 0$ and there is no non-zero solution.

Case II: $2 - \lambda = 0$. The d.e. is $y'' = 0$. The solution is

$$\begin{aligned} y &= c_1 x + c_2 \\ y' &= c_1 \end{aligned}$$

From $y'(0) = 0$,

$$c_1 = 0.$$

Then $y'(\pi) = 0$ and c_2 is arbitrary. Hence we have an eigenvalue of $\lambda = 2$ and will label this as $\lambda_0 = 2$ with the corresponding eigenfunction labeled as $y_0 = c_0$.

Case III. $2 - \lambda < 0$. Let's write $2 - \lambda = -\mu^2$. Hence $r = \pm\sqrt{2 - \lambda} = \pm\mu i$ and the solution to the d.e. is

$$\begin{aligned} y &= c_1 \cos \mu x + c_2 \sin \mu x \\ y' &= \mu(-c_1 \sin \mu x + c_2 \cos \mu x) \end{aligned}$$

From $y'(0) = 0$,

$$c_2 = 0.$$

From $y'(\pi) = 0$,

$$-\mu c_1 \sin \mu\pi = 0.$$

For a non-zero solution, we need to have $\sin \mu\pi = 0$. So eigenvalues and eigenfunctions come from

$$\begin{aligned} \mu_n &= n & n &= 1, 2, 3, \dots \\ \lambda_n &= 2 + \mu^2 \\ &= 2 + n^2 & n &= 1, 2, 3, \dots \\ y_n &= c_n \cos nx & n &= 1, 2, 3, \dots \end{aligned}$$

For the ease of the grader, we can combine cases II and III and summarize the results as

$$\begin{aligned} \text{eigenvalues : } \lambda_n &= 2 + n^2 & n &= 0, 1, 2, 3, \dots \\ \text{eigenfunctions : } y_n &= c_n \cos nx & n &= 0, 1, 2, 3, \dots \end{aligned}$$

5. (a) (15 pts.) Find the Fourier *sine* series for the function

$$f(x) = x \text{ on } 0 < x < 1$$

Solution:

$$f(x) = \sum_1^{\infty} \alpha_k \sin \frac{k\pi x}{L}$$

where

$$\alpha_k = \frac{2}{L} \int_0^L f(x) \sin \frac{k\pi x}{L} dx, \quad k = 1, 2, 3, \dots$$

Here $L = 1$ so

$$f(x) = \sum_1^{\infty} \alpha_k \sin(k\pi x)$$

where

$$\alpha_k = 2 \int_0^1 f(x) \sin(k\pi x) dx, \quad k = 1, 2, 3, \dots$$

Thus

$$\begin{aligned} \alpha_k &= 2 \int_0^1 x \sin(k\pi x) dx = 2 \left[\frac{1}{(k\pi)^2} (\sin k\pi x - k\pi x \cos k\pi x) \right]_0^1 = \\ &= -2 \left[\frac{1}{k\pi} \cos k\pi \right] = \frac{2}{k\pi} (-1)^{k+1} \quad k = 1, 2, 3, \dots \end{aligned}$$

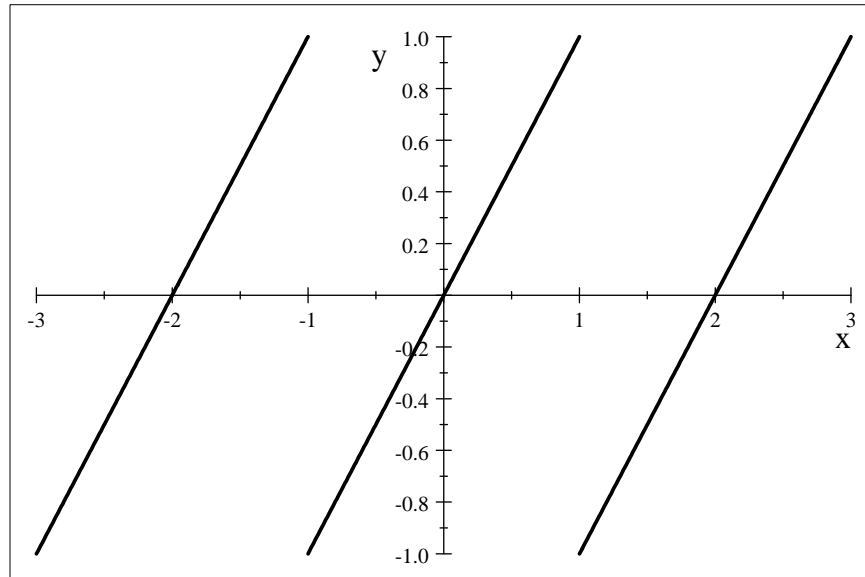
Thus

$$f(x) = \frac{2}{\pi} \sum_1^{\infty} \frac{(-1)^{k+1}}{k} \sin(k\pi x)$$

(b) (10 pts.) Sketch the graph of the function represented by the Fourier *sine* series in 5 (a) on $-3 < x < 3$.

Solution:

1



6 (25 pts.)

$$\text{PDE} \quad u_{xx} - 16u_{tt} = 0$$

$$\text{BCs} \quad u(0, t) = 0 \quad u_x(1, t) = 0$$

$$\text{IC} \quad u(x, 0) = -6 \sin\left(\frac{3\pi x}{2}\right) + 13 \sin\left(\frac{11\pi x}{2}\right)$$

$$\text{IC} \quad u_t(x, 0) = 0$$

You must derive the solution. Your solution should not have any arbitrary constants in it. Show **all** steps.

Solution: Let $u(x, t) = X(x)T(t)$. Then the PDE implies

$$X''T = 16XT''$$

or

$$\frac{X''}{X} = 16\frac{T''}{T} = -\lambda^2$$

since we will need sines and cosines in the X part of the solution.

Thus

$$X'' + \lambda^2 X = 0$$

$$T'' + \frac{\lambda^2}{16} T = 0$$

The BCs are

$$X(0) = X'(1) = 0$$

$$X(x) = a_n \sin \lambda x + b_n \cos \lambda x$$

$X(0) = 0$ implies that $b_n = 0$, so

$$X(x) = a_n \sin \lambda x$$

$$X'(x) = a_n \lambda \cos \lambda x$$

so

$$X'(1) = a_n \lambda \cos \lambda = 0$$

Hence $\lambda = \frac{2n+1}{2}\pi$, $n = 0, 1, 2, \dots$ and

$$X_n(x) = A_n \sin\left(\frac{2n+1}{2}\pi x\right) \quad n = 0, 1, 2, \dots$$

Also

$$T'' + \frac{\lambda^2}{16} T = T'' + \frac{(2n+1)^2 \pi^2}{64} T = 0$$

$$T_n(t) = c_n \sin\left(\frac{2n+1}{8}\pi t\right) + d_n \cos\left(\frac{2n+1}{8}\pi t\right)$$

$u_t(x, 0) = 0$ implies that $c_n = 0$ and

$$T_n(t) = d_n \cos\left(\frac{2n+1}{8}\pi t\right)$$

Thus

$$u_n(x, t) = B_n \sin\left(\frac{2n+1}{2}\pi x\right) \cos\left(\frac{2n+1}{8}\pi t\right) \quad n = 0, 1, 2, \dots$$

Let

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2}\right)\pi x \cos\left(\frac{2n+1}{8}\right)\pi t$$

$$u(x, 0) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2}\right)\pi x = -6 \sin\left(\frac{3\pi x}{2}\right) + 13 \sin\left(\frac{11\pi x}{2}\right)$$

Therefore $B_1 = -6$, $B_5 = 13$ and $B_n = 0$ for $n \neq 1, 5$ so

$$u(x, t) = -6 \sin\left(\frac{3\pi x}{2}\right) \cos\left(\frac{3\pi}{8}\right)t + 13 \sin\left(\frac{11\pi x}{2}\right) \cos\left(\frac{11\pi}{8}\right)t$$

7. (a) (10 pts.) Solve

$$x^2 y'' + 3xy' + 2y = 0$$

Solution: This is a Cauchy-Euler equation with $p = 3$ and $q = 2$ so the indicial equation is

$$m^2 + m(p - 1) + q = m^2 + 2m + 2$$

Thus

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = -1 \pm i$$

Thus we set $a = -1$ and $b = 1$ in the formula

$$y_h = x^a [A \cos(b \ln x) + B \sin(b \ln x)].$$

and get

$$y_h = x^{-1} [A \cos(\ln x) + B \sin(\ln x)]$$

(b) (15 pts.) Find the first 5 nonzero terms of the power series solution about $x = 0$ for the DE:

$$y'' - 4xy' + 4y = 0$$

Be sure to give the recurrence relation.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

so

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

.

The differential equation \Rightarrow

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - 4 \sum_{n=1}^{\infty} a_n n x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

Combining the second and third summations we have

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + 4a_0 + 4 \sum_{n=1}^{\infty} a_n (1-n) x^n = 0$$

Shifting the first series by letting $n - 2 = k$ or $n = k + 2$ we have

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^k + 4a_0 + 4 \sum_{n=1}^{\infty} a_n (1-n) x^n = 0$$

or after replacing k and n by m and combining the series

$$4a_0 + 2a_2 + \sum_{m=1}^{\infty} [a_{m+2} (m+2)(m+1) + 4a_m (1-m)] x^m = 0$$

Thus

$$a_2 = -2a_0$$

and the recurrence relation is

$$a_{m+2} = \frac{4(m-1)}{(m+2)(m+1)}a_m \quad m = 1, 2, 3, \dots$$

Thus

$$a_3 = 0$$

and hence

$$a_{2n+1} = 0 \quad \text{for } n = 1, 2, 3, \dots$$

Also

$$a_4 = \frac{4}{4(3)}a_2 = -\frac{2}{3}a_0$$

$$a_6 = \frac{4(3)}{6(5)}a_4 = \frac{2}{5}a_4 = -\frac{4}{15}a_0$$

Thus

$$y = a_1x + a_0\left(1 - 2x^2 - \frac{2}{3}x^4 - \frac{4}{15}x^6 - \dots\right)$$

SNB check: $y'' - 4xy' + 4y = 0$, Series solution is:

$$\left\{y(0) + xy'(0) - 2x^2y(0) - \frac{2}{3}x^4y(0) - \frac{4}{15}x^6y(0) + O(x^7)\right\}$$

8 (a) (10 pts.) Solve

$$\frac{dy}{dx} + y = \frac{e^x}{y} \quad y(0) = 1$$

Solution. We rewrite the equation as

$$\frac{dy}{dx} + y = e^x y^{-1}$$

so this is a Bernoulli equation. Multiplying by y we have

$$yy' + y^2 = e^x$$

Let $z = y^2$ so that $z' = 2yy'$ and the DE becomes

$$\frac{1}{2}z' + z = e^x$$

or

$$z' + 2z = 2e^x$$

The integrating factor is $e^{\int 2dx} = e^{2x}$. Multiplying the DE by this we have

$$\frac{d(ze^{2x})}{dx} = 2e^{3x}$$

so

$$ze^{2x} = \frac{2}{3}e^{3x} + c$$

and

$$z = y^2 = \frac{2}{3}e^x + ce^{-2x}$$

The initial condition implies $c = \frac{1}{3}$ so

$$y^2 = \frac{2}{3}e^x + \frac{1}{3}e^{-2x}$$

(b) (15 pts.) Find

$$\mathcal{L}^{-1} \left\{ \frac{25}{s^3(s^2 + 4s + 5)} \right\}$$

Solution: The partial fractions breakdown is

$$\frac{25}{s^3(s^2 + 4s + 5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 4s + 5}$$

Multiplying both sides by s^3 and setting $s = 0$ yields

$$5 = C$$

so

$$\frac{25}{s^3(s^2 + 4s + 5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{5}{s^3} + \frac{Ds + E}{s^2 + 4s + 5}$$

Putting everything on the right over the common denominator $s^3(s^2 + 4s + 5)$ and equating the numerators on both sides leads to

$$25 = As^2(s^2 + 4s + 5) + Bs(s^2 + 4s + 5) + 5(s^2 + 4s + 5) + Ds^4 + Es^3$$

Thus

$$\begin{aligned}s^4 : \quad A + D &= 0 \\s^3 : \quad 4A + B + E &= 0 \\s^2 : \quad 5A + 4B + 5 &= 0 \\s : \quad 5B + 20 &= 0\end{aligned}$$

Hence $B = -4, A = \frac{11}{5}, D = -\frac{11}{5}$

$$E = -4\left(\frac{11}{5}\right) + 4 = -\frac{24}{5}$$

$$\begin{aligned}\frac{25}{s^3(s^2 + 4s + 5)} &= \frac{1}{5} \left[\frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11s + 24}{s^2 + 4s + 5} \right] \\&= \frac{1}{5} \left[\frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11s + 24}{(s + 2)^2 + 1} \right] \\&= \frac{1}{5} \left[\frac{11}{s} - \frac{20}{s^2} + \frac{25\left(\frac{2!}{2!}\right)}{s^3} - \frac{11(s + 2)}{(s + 2)^2 + 1} - \frac{2}{(s + 2)^2 + 1} \right]\end{aligned}$$

Hence

$$\mathcal{L}^{-1} \left\{ \frac{25}{s^3(s^2 + 4s + 5)} \right\} = \frac{1}{5} \left(11 - 20t + \frac{25}{2}t^2 - 11e^{-2t} \cos t - 2e^{-2t} \sin t \right)$$

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\frac{1}{s-a}$		$s > a$
$\sin bt$	$\frac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} x + C$
$\int \cos^2 x dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} x + C$
$\int x \cos bx dx = \frac{1}{b^2} (\cos bx + bx \sin bx) + C$
$\int x \sin bx dx = \frac{1}{b^2} (\sin bx - bx \cos bx) + C$
$\int \left(\frac{e^{-t}}{1+e^{-t}} \right) dt = -\ln(1+e^{-t}) + C$
$\int \left(\frac{e^{-2t}}{1+e^{-t}} \right) dt = \ln(1+e^{-t}) - e^{-t} + C$
$\int x e^{ax} dx = \frac{1}{a^2} (a x e^{ax} - e^{ax}) + C$