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Ma 221 **Final Exam Solutions** **5/14/13**

1. Solve

(a) (8 pts)

$$\frac{dy}{dx} = \frac{e^{x+y}}{y-1} \qquad y(0) = 0$$

Solution: The equation is separable.

$$(y-1)e^{-y}dy = e^{x}dx$$
$$\int (y-1)e^{-y}dy = \int e^{x}dx$$
$$\int (ye^{-y} - e^{-y})dy = \int e^{x}dx$$
$$-ye^{-y} - e^{-y} + e^{-y} = e^{x} + c$$

The last step comes from the integral table. We use the initial condition $y(0) = 0$ to find the constant. $c = -1$.

So the solution is given implicitly by

$$-ye^{-y} = e^{x} - 1$$

or

$$e^{x} + ye^{-y} = 1$$

(b) (7 pts) Solve

$$y\frac{dx}{dy} + 2x = 5y^{3}$$

Solution: The d.e. is linear. We write it in the standard form.

$$\frac{dx}{dy} + \frac{2}{y}x = 5y^{2}$$

So the integrating factor is

$$e^{\int \frac{2}{y}dy} = e^{2\ln y} = e^{\ln y^{2}} = y^{2}.$$

Multiplication by this factor and integrating gives the solution.

$$y^{2}\frac{dx}{dy} + 2yx = 5y^{4}$$
$$\frac{d}{dy}\left(y^{2}x\right) = 5y^{4}$$
$$y^{2}x = y^{5} + c$$
$$x = y^{3} + cy^{-2}.$$

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1 (c) (10 pts) Solve

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}, \qquad y(1) = 4$$

Solution: This is a Bernoulli d.e. We begin by putting the equation into the standard form.

$$\frac{dy}{dx} - \frac{y}{x} = xy^{-1}$$

A little algebra will lead to the substitution $v = y^2$ which will give a linear d.e.

$$y\frac{dy}{dx} - \frac{1}{x}y^2 = x$$

$$\frac{1}{2}\frac{dv}{dx} - \frac{1}{x}v = x$$

$$\frac{dv}{dx} - \frac{2}{x}v = 2x$$

The integrating factor is

$$e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = e^{\ln(x^{-2})} = x^{-2}.$$

We multiply by this factor and integrate.

$$\frac{1}{x^2}\frac{dv}{dx} - \frac{2}{x^3}v = \frac{2}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}v\right) = \frac{2}{x}$$

$$\frac{1}{x^2}v = 2\ln x + c$$

$$v = 2x^2\ln x + cx^2$$

$$y^2 = 2x^2\ln x + cx^2$$

Finally, we use the initial condition to obtain the constant.

$$4^2 = 2\ln 1 + c$$

$$c = 16$$

Thus the solution is given implicitly by

$$y^2 = 2x^2\ln x + 16x^2$$

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2. (a) (12 pts) Find a general solution of

$$y'' + 4y' + 5y = 10 + 8\sin t$$

Solution: First, we solve the corresponding homogeneous equation by finding the roots of the characteristic polynomial.

$$\begin{aligned} r^2 + 4r + 5 &= 0 \\ r^2 + 4r + 4 + 1 &= 0 \\ r^2 + 4r + 4 &= -1 \\ (r+2)^2 &= -1 \\ r + 2 &= \pm i \\ r &= -2 \pm i \\ y_h &= c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t \end{aligned}$$

Next, we solve

$$y'' + 4y' + 5y = 10$$

The solution has the form

$$y = A$$

Substitution gives

$$\begin{aligned} 5A &= 10 \\ A &= 2 \\ y_{p_1} &= 2 \end{aligned}$$

To solve

$$y'' + 4y' + 5y = 8\sin t$$

we seek a solution of the form

$$\begin{aligned} y &= A\cos t + B\sin t \\ y' &= -A\sin t + B\cos t \\ y'' &= -A\cos t - B\sin t \end{aligned}$$

Substitution gives

$$-A\cos t - B\sin t + 4(-A\sin t + B\cos t) + 5(A\cos t + B\sin t) = 8\sin t$$

Equating the coefficients of $\cos t$ and $\sin t$, we obtain two equations.

$$\begin{aligned} -A + 4B + 5A &= 0 \\ -B - 4A + 5B &= 8 \end{aligned}$$

I.e.

$$\begin{aligned} 4A + 4B &= 0 \\ -4A + 4B &= 8 \\ B &= 1 \\ A &= -1 \\ y_{p_2} &= -\cos t + \sin t \end{aligned}$$

Finally, we put it all together.

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$$\begin{aligned} y &= y_h + y_{p_1} + y_{p_2} \\ &= c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t + 2 - \cos t + \sin t \end{aligned}$$

SNB check $y'' + 4y' + 5y = 10 + 8\sin t$, Exact solution is:
$\left\{\sin t - \cos t + C_2(\cos t)e^{-2t} - C_3(\sin t)e^{-2t} + 2\right\}$

2(b) (13 pts.) Find a general solution of

$$y'' + 16y = \tan 4t$$

Solution: The characteristic equation is

$$\begin{aligned} r^2 + 16 &= 0 \\ r^2 &= -16 \\ r &= \pm 4i \end{aligned}$$

Hence, the solution of the homogeneous equation is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 \cos 4t + c_2 \sin 4t. \end{aligned}$$

To find a particular solution of the given equation, we use variation of parameters. The form of the solution is

$$\begin{aligned} y_h &= v_1 y_1 + v_2 y_2 \\ &= v_1 \cos 4t + v_2 \sin 4t \end{aligned}$$

The derivatives of the unknowns satisfy

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= \tan 4t \end{aligned}$$

Substituting the solutions from the homogeneous equation, we obtain

$$\begin{aligned} (\cos 4t)v_1' + (\sin 4t)v_2' &= 0 \qquad \text{A} \\ (-4\sin 4t)v_1' + (4\cos 4t)v_2' &= \tan 4t \end{aligned}$$

Multiplying the first equation by $4\sin 4t$ and the second equation by $\cos 4t$ gives

$$\begin{aligned} (4\sin 4t)(\cos 4t)v_1' + 4\left(\sin^2 4t\right)v_2' &= 0 \\ (\cos 4t)(-4\sin 4t)v_1' + \left(4\cos^2 4t\right)v_2' &= \sin 4t \end{aligned}$$

Add these to obtain

$$\begin{aligned} 4\left(\sin^2 4t + \cos^2 4t\right)v_2' &= \sin 4t \\ v_2' &= \frac{1}{4}\sin 4t \\ v_2 &= -\frac{1}{16}\cos 4t + c_2 \end{aligned}$$

Substituting the value of v_2' into equation A gives

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$$
\begin{aligned}
(\cos 4t)v_1' + (\sin 4t)\left(\frac{1}{4}\sin 4t\right) &= 0 \\
v_1' &= \frac{1}{4}\,\frac{\sin^2 4t}{\cos 4t} \\
&= \frac{1}{4}\,\frac{1-\cos^2 4t}{\cos 4t} \\
&= \frac{1}{4}(\sec 4t - \cos 4t) \\
v_1 &= \frac{1}{4}\int(\sec 4t - \cos 4t)dt \\
&= \frac{1}{16}(\ln|\sec 4t + \tan 4t| - \sin 4t) + c_1
\end{aligned}
$$

Finally, we combine everything to display the solution.

$$
\begin{aligned}
y &= v_1 \cos 4t + v_2 \sin 4t \\
&= \left[\frac{1}{16}(\ln|\sec 4t + \tan 4t| - \sin 4t) + c_1\right]\cos 4t + \left(-\frac{1}{16}\cos 4t + c_2\right)\sin 4t
\end{aligned}
$$

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3. (a) (10 pts.) Let $F(s) = \mathcal{L}\{f(t)\}$. Given that $f(t)$ is continuous and of exponential order α, use the definition of the Laplace Transform to show that

$$\mathcal{L}\{tf(t)\} = -\frac{dF}{ds}.$$

Solution:

$$\begin{aligned}
F(s) &= \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt \\
\frac{dF}{ds} &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt \\
&= \int_0^\infty \frac{d}{ds}(e^{-st}f(t))dt \\
&= \int_0^\infty (-te^{-st}f(t))dt \\
&= \int_0^\infty e^{-st}(-tf(t))dt \\
&= -\int_0^\infty e^{-st}(tf(t))dt \\
&= -\mathcal{L}\{tf(t)\}
\end{aligned}$$

Multiplication by –1 gives the desired result.

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(b) (15 pts.) Solve using Laplace Transforms:

$$y'' - 4y' + 4y = 3te^{2t} \qquad y(0) = 4, \;\; y'(0) = 2$$

Solution: We take the Laplace Transform, using the following notation and formulae.

$$\begin{aligned} Y &= \mathcal{L}\{y\} \\ \mathcal{L}\left\{y'\right\} &= sY - y(0) \\ \mathcal{L}\left\{y''\right\} &= s^2Y - sy(0) - y'(0) \end{aligned}$$

The transform of the d.e. is

$$\begin{aligned} s^2Y - sy(0) - y'(0) - 4[sY - y(0)] + 4Y &= \frac{3}{(s-2)^2} \\ \left(s^2 - 4s + 4\right)Y - 4s - 2 + 16 &= \frac{3}{(s-2)^2} \\ \left(s^2 - 4s + 4\right)Y &= 4s - 14 + \frac{3}{(s-2)^2} \\ Y &= \frac{4s-14}{(s-2)^2} + \frac{3}{(s-2)^4} \\ &= \frac{4(s-2)-6}{(s-2)^2} + \frac{3}{(s-2)^4} \\ &= \frac{4}{(s-2)} + \frac{-6}{(s-2)^2} + \frac{3}{(s-2)^4} \end{aligned}$$

Obtaining the inverse Laplace Transforms from the table yields the solution.

$$y = 4e^{2t} - 6te^{2t} + \frac{3}{3!}t^3e^{2t} = 4e^{2t} - 6te^{2t} + \frac{1}{2}t^3e^{2t}$$

SNB Check:

$$\begin{aligned} y'' - 4y' + 4y &= 3te^{2t} \\ y(0) &= 4 \\ y'(0) &= 2 \end{aligned}$$

, Exact solution is: $\left\{4e^{2t} - 6te^{2t} + \frac{1}{2}t^3e^{2t}\right\}$

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4.) a.) (10 pts.) Use separation of variables, $u(x,t) = X(x)T(t)$, to find two ordinary differential equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$e^{2x+3t}\frac{\partial^2 u}{\partial x\partial t} - (x+4)^5(t^2+7)^8\frac{\partial^2 u}{\partial x^2} = 0$$

Note: Do **not** solve these ordinary differential equations.

Solution: We substitute the given expression and then try to rewrite the equation with everything involving x on one side of the equation and everything involving t on the other side.

$$\begin{aligned}
u(x,t) &= X(x)T(t)\\
\frac{\partial^2 u}{\partial x\partial t} &= X'(x)T'(t)\\
\frac{\partial^2 u}{\partial x^2} &= X''(x)T(t)\\
e^{2x+3t}X'(x)T'(t) - (x+4)^5(t^2+7)^8X''(x)T(t) &= 0\\
e^{2x}e^{3t}X'(x)T'(t) &= (x+4)^5(t^2+7)^8X''(x)T(t)\\
\frac{e^{3t}}{(t^2+7)^8}\frac{T'(t)}{T(t)} &= (x+4)^5e^{-2x}\frac{X''(x)}{X'(x)}
\end{aligned}$$

Since we have a function only of t equal to a function only of x, the result must be a constant. Any name is acceptable, so I choose K and write the two resulting differential equations..

$$\begin{aligned}
\frac{e^{3t}}{(t^2+7)^8}\frac{T'(t)}{T(t)} &= (x+4)^5e^{-2x}\frac{X''(x)}{X'(x)} = K\\
e^{3t}T'(t) &= K(t^2+7)^8T(t)\\
(x+4)^5e^{-2x}X''(x) &= KX'(x)
\end{aligned}$$

b.) (15 pts.) Find

$$\mathcal{L}^{-1}\left\{\frac{4s^2+13s+19}{(s-1)\left(s^2+4s+13\right)}\right\}$$

Solution: We use partial fractions to break this into the kind of expressions in the table of Laplace Transforms.

$$\begin{aligned}
\frac{4s^2+13s+19}{(s-1)\left(s^2+4s+13\right)} &= \frac{4s^2+13s+19}{(s-1)\left(s^2+4s+4+9\right)}\\
&= \frac{4s^2+13s+19}{(s-1)\left[(s+2)^2+9\right]}\\
&= \frac{A}{s-1} + \frac{B(s+2)+C}{(s+2)^2+9}
\end{aligned}$$

Multiplication by the common denominator gives

$$4s^2+13s+19 = A\left[(s+2)^2+9\right] + [B(s+2)+C](s-1)$$

The choice $s = 1$ yields

$$4+13+19 = 36 = A\left[3^2+9\right]$$

$$A = 2$$

Set $s = -2$ to obtain

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$$4 \cdot 4 - 26 + 19 = A \cdot 9 + C \cdot (-3)$$
$$9 = 18 - 3C$$
$$C = 3$$

Now, $s = 0$ gives

$$19 = A[4+9] + (2B + C)(-1)$$
$$19 = 26 - 2B - 3$$
$$2B = 4$$
$$B = 2$$

So

$$\frac{4s^2 + 13s + 19}{(s-1)\left(s^2 + 4s + 13\right)} = \frac{A}{s-1} + \frac{B(s+2) + C}{(s+2)^2 + 9}$$
$$= \frac{2}{s-1} + \frac{2(s+2)}{(s+2)^2 + 3^2} + \frac{3}{(s+2)^2 + 3^2}$$

$$\mathcal{L}^{-1}\left\{\frac{4s^2 + 13s + 19}{(s-1)\left(s^2 + 4s + 13\right)}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2 + 3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2 + 3^2}\right\}$$
$$= 2e^t + 2e^{-2t}\cos 3t + e^{-2t}\sin 3t$$

SNB check $\frac{4s^2+13s+19}{(s-1)(s^2+4s+4+9)} = \frac{2}{s-1} + \frac{2s+7}{s^2+4s+13}$

$\frac{4s^2+13s+19}{(s-1)(s^2+4s+4+9)}$, Is Laplace transform of $2e^t + 2e^{-2t}\left(\cos 3t + \frac{1}{2}\sin 3t\right)$

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5. (a) (15 pts.) Find the first five non-zero terms of the Fourier *cosine* series for the function

$$f(x) = x \text{ on } 0 < x < \pi$$

Solution:

$$f(x) = \frac{a_0}{2} + \sum_1^\infty \alpha_k \cos\frac{n\pi x}{L}$$

where

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, 3, \ldots$$

Here $L = \pi$ so

$$f(x) = \frac{a_0}{2} + \sum_1^\infty \alpha_n \cos nx$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos nx dx, \quad n = 0, 1, 2, 3, \ldots$$

$$a_0 = \frac{2}{\pi}\int_0^\pi x dx = \pi$$

$\int x\cos nx dx = \frac{1}{n^2}(\cos nx + nx\sin nx) =$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx dx, \quad n = 1, 2, 3, \ldots \\
&= \frac{2}{\pi n^2}(\cos nx + nx\sin nx)|_0^\pi \quad n = 1, 2, 3, \ldots \\
&= \frac{2}{\pi n^2}(\cos n\pi - 1) = \frac{2}{\pi n^2}((-1)^n - 1) \quad n = 1, 2, 3, \ldots
\end{aligned}$$

Thus

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \frac{1}{49}\cos 7x + \cdots\right]$$

(b) (10 pts.) Sketch the graph of the function represented by the Fourier *cosine* series in 5 (a) on $-\pi < x < 3\pi$.

Solution:

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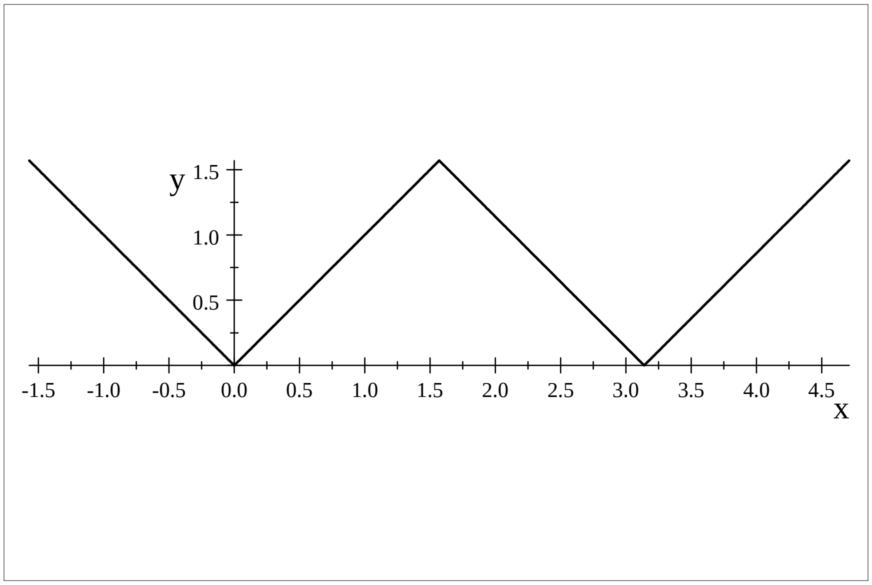

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6 (25 pts.)

$$\text{PDE} \qquad u_{xx} = u_t$$
$$\text{BCs} \quad u(0,t) = 0 \qquad u(\pi,t) = 0$$
$$\text{ICs} \quad u(x,0) = -17\sin 5x$$

You must derive the solution. Your solution should not have any arbitrary constants in it. Show **all** steps.

Solution: $u(x,t) = X(x)T(t)$

$$X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda^2$$

Therefore

$$X'' + \lambda^2 X = 0 \quad X(0) = X(\pi) = 0$$

$$X(x) = c_1 \sin\lambda x + c_2\cos\lambda x$$

$X(0) = 0 \Rightarrow c_2 = 0,\;\; X(\pi) = c_1\sin(\lambda\pi) = 0$ so

$$\lambda = n, \quad n = 1,2,3,\ldots$$

and

$$X_n(x) = c_n\sin(nx)$$

The equation for T is

$$T' + \lambda^2 T = T' + n^2 T = 0 \Rightarrow T_n(t) = a_n e^{-n^2 t}$$

$$u_n(x,t) = X_n(x)T_n(t) = A_n e^{-n^2 t}\sin nx,\;\; n = 1,2,\ldots$$

so

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2 t} A_n \sin nx$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n\sin nx = -17\sin 5x$$

Therefore $A_5 = -17, A_n = 0, n \neq 5$ so

$$u(x,t) = -17e^{-25t}\sin 5x$$

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7. (a) (10 pts.) Find a general solution of

$$y'' - 4y' + 4y = 3e^{2t} + 8$$

Solution: The characteristic equation is

$$p(r) = r^2 - 4r + 4 = (r-2)^2 = 0$$

Thus $r = 2$ is a repeated root and the homogeneous solution is

$$y_h = c_1 e^{2t} + c_2 t e^{2t}$$

Since $p(2) = p'(2) = 0$ a particular solution for $3e^{2t}$ is

$$y_{p_1} = \frac{kt^2 e^{\alpha t}}{p''(\alpha)} = \frac{3}{2} t^2 e^{2t}$$

By inspection we see that a particular solution for 8 is 2. Thus

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^{2t} + c_2 t e^{2t} + \frac{3}{2} t^2 e^{2t} + 2$$

SNB Check: $y'' - 4y' + 4y = 3e^{2t} + 8$, Exact solution is: $C_1 e^{2t} + C_2 t e^{2t} + \frac{3}{2} t^2 e^{2t} + 2$

(b) (15 pts.) Find the power series solution to

$$y'' - xy' + 2y = 0$$

near $x = 0$. Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first six nonzero terms of the solution.

Solution:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

The DE implies

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=1}^{\infty} a_n (n) x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$2a_0 + \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n (2-n) x^n = 0$$

We shift the second sum by letting $k - 2 = n$ or $k = n + 2$ and get

$$2a_0 + \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{k=3}^{\infty} a_{k-2} (2-k+2) x^{k-2} = 0$$

Replacing n and k by m and combining we have

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$$2a_0 + +(2)(1)a_2 + \sum_{m=3}^{\infty}\{a_m(m)(m-1) + a_{m-2}(4-m)\}x^{m-2} = 0$$

so

$$2a_0 + (2)(1)a_2 = 0 \Rightarrow a_2 = -a_0$$

$$a_m(m)(m-1) + a_{m-2}(4-m) = 0 \quad m = 3, 4, \ldots$$

or

$$a_m = -\left(\frac{4-m}{m(m-1)}\right)a_{m-2} \quad m = 3, 4, \ldots$$

Then

$$a_3 = -\frac{1}{3(2)}a_1$$

$$a_4 = 0$$

$$a_5 = \frac{1}{5(4)}a_3 = -\frac{1}{5!}a_1$$

$$a_6 = 0$$

$$a_7 = \frac{3}{7(6)}a_5 = -\frac{3}{7!}a_1$$

Thus

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots = a_0\left[1 - x^2\right] + a_1\left[x - \frac{1}{3!}x^3 - \frac{1}{5!}x^5 - \frac{3}{7!}x^7 + \cdots\right]$$

SNB Check: $y'' - xy' + 2y = 0$, Series solution is: $y(x) = y(0) + xy'(0) -$
$y(0)x^2 - \frac{1}{6}y'(0)x^3 - \frac{1}{120}y'(0)x^5 - \frac{1}{1680}y'(0)x^7,$

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8 (a) (15 pts.) Find the eigenvalues and eigenfunctions for

$$x^2y'' + xy' + \lambda y = 0 \qquad y(1) = y(e^\pi) = 0$$

Be sure to consider the cases $\lambda < 0, \lambda = 0,$ and $\lambda > 0.$

Solution:

Note that this is an Euler equation. Since $p = 1, q = \lambda$ the indicial equation is

$$r^2 + (p-1)r + q = r^2 + \lambda = 0$$

I. $\lambda < 0.$ Let $\lambda = -\alpha^2$ where $\alpha \neq 0.$ Then the indicial equation is

$$r^2 - \alpha^2 = 0$$

so $m = \pm\alpha.$ Thus

$$y = c_1 x^\alpha + c_2 x^{-\alpha}$$

The boundary conditions imply

$$c_1 + c_2 = 0$$
$$c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = 0$$

This leads to $c_1 = c_2 = 0,$ so $y = 0$ and there are no negative eigenvalues.

II. $\lambda = 0.$ The DE for this case is

$$x^2 y'' + xy' = 0$$

or

$$xy'' + y' = 0$$

Let $v = y'$ so we have

$$xv' + v = 0$$
$$(xv)' = 0$$

Then $v = \frac{c_1}{x},$ and $y = c_1 \ln x + c_2.$ The BCs lead to

$$c_1 + c_2 = 0$$
$$c_1 \ln e^\pi + c_2 = 0$$

or

$$c_1 + c_2 = 0$$
$$\pi c_1 + c_2 = 0$$

Again we have $c_1 = c_2 = 0,$ so $\lambda = 0$ is not an eigenvalue.

III. $\lambda > 0.$ Let $\lambda = \beta^2$ where $\beta \neq 0.$ The indicial equation is

$$r^2 + \beta^2 = 0$$

so

$$r = \pm\beta i$$

Thus

$$y = c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)$$

The BCs imply

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$$c_1 = 0 \text{ and } c_2 \sin \beta\pi = 0$$
$$\text{Thus} \quad c_2 \text{ is arbitrary}$$
$$\sin(\beta\pi) = 0 \Rightarrow \beta = n \text{ for } n = 1, 2, \ldots$$

Eigenvalues:

$$\lambda_n = \beta^2 = n^2 \; n = 1, 2, \ldots$$

Eigenfunctions:

$$y_n = c_n \sin(n \ln x)$$

8(b) (10 pts.) Show that if the equation

$$\left(\frac{y^2}{2} + 2ye^x\right)dx + (y + e^x)dy = 0$$

is multiplied by e^x, then the resulting equation is exact and then solve this equation.
Solution: Multiplying by e^x we have

$$\left(\frac{y^2e^x}{2} + 2ye^{2x}\right)dx + \left(ye^x + e^{2x}\right)dy = 0$$

so

$$M_y = \frac{\partial\left(\frac{y^2e^x}{2} + 2ye^{2x}\right)}{\partial y} = ye^x + 2e^x = N_x = \frac{\partial\left(ye^x + e^{2x}\right)}{\partial x}$$

so the multiplied equation is now exact. Hence there exists $f(x,y)$ such that

$$f_x = M = \frac{y^2e^x}{2} + 2ye^{2x}$$
$$f_y = N = ye^x + e^{2x}$$

Starting with f_y and integrating with respect to y we have

$$f(x,y) = \frac{y^2}{2}e^x + ye^{2x} + g(x)$$

so

$$f_x = \frac{y^2}{2}e^x + 2ye^{2x} + g'(x) = M = \frac{y^2e^x}{2} + 2ye^{2x}$$

Thus $g'(x) = 0$ so $g(x) = k$ where k is a constant. Thus the solution is given by

$$\frac{y^2}{2}e^x + ye^{2x} = C$$

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Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$		$s > a$
$\sin bt$	$\dfrac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int x \cos bx dx = \frac{1}{b^2}(\cos bx + bx \sin bx) + C$
$\int \tan ax dx = -\frac{1}{a}\ln(\cos ax) + C$
$\int \sec ax dx = \frac{1}{a}\ln\lvert\sec ax + \tan ax\rvert + C$
$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax + C$
$\int x \sin bx dx = \frac{1}{b^2}(\sin bx - bx \cos bx) + C$
$\int \left(\frac{e^{-t}}{1+e^{-t}}\right) dt = -\ln(1+e^{-t}) + C$
$\int \left(\frac{e^{-2t}}{1+e^{-t}}\right) dt = \ln(1+e^{-t}) - e^{-t} + C$
$\int x e^{ax} dx = \frac{1}{a^2}(axe^{ax} - e^{ax}) + C$