Name__________________________ Lecturer _______________________

Ma 221 **Final Exam Solutions** **5/6/14**

Print Name: _________________________________ **Lecture Section**: _________

1.(a) (8 pts)Solve

$$xy' + 2y = 8x^2 + 6x \quad y(1) = 2.$$

Solution: The d.e. is linear. We put it into standard form.

$$y' + \frac{2}{x}y = 8x + 6$$

Next, we find the integrating factor, μ, multiply the d.e. by the integrating factor and integrate.

$$\begin{aligned} \mu &= \exp\left(\int \frac{2}{x}dx\right) = e^{2\ln x} \\ &= e^{\ln(x^2)} = x^2 \end{aligned}$$

$$\begin{aligned} x^2y' + 2xy &= 8x^3 + 6x^2 \\ \frac{d}{dx}\left(x^2y\right) &= 8x^3 + 6x^2 \\ x^2y &= 2x^4 + 2x^3 + c \end{aligned}$$

Finally, we use the initial condition to find the constant and display the answer.

$$\begin{aligned} 1^2 \cdot 2 &= 2 \cdot 1^4 + 2 \cdot 1^3 + c \\ c &= -2 \\ x^2y &= 2x^4 + 2x^3 - 2 \\ y &= 2x^2 + 2x - \frac{2}{x^2} \end{aligned}$$

(b) (7 pts) Solve

$$\left(2xy^3 + 6x\right)dx + \left(3x^2y^2 + 10y\right)dy = 0.$$

Solution: Clearly the d.e. is not separable or linear. We test for an exact d.e.

$$\begin{aligned} M &= 2xy^3 + 6x & N &= 3x^2y^2 + 10y \\ M_y &= 6xy^2 & N_x &= 6xy^2 \end{aligned}$$

$M_y = N_x$. Hence, the d.e. is exact. We proceed to find a function, F, such that

$$dF = F_x + F_y = \left(2xy^3 + 6x\right)dx + \left(3x^2y^2 + 10y\right)dy.$$

From

$$\begin{aligned} F_x &= 2xy^3 + 6x, \\ F &= \int\left(2xy^3 + 6x\right)\partial x \\ &= x^2y^3 + 3x^2 + g(y). \end{aligned}$$

Now, we must have

$$F_y = 3x^2y^2 + g'(y) = N = 3x^2y^2 + 10y$$

$$\begin{aligned} g'(y) &= 10y \\ g(y) &= 5y^2. \end{aligned}$$

Finally, the solution is

$$F = x^2y^3 + 3x^2 + 5y^2 = c.$$

Name___________________________ Lecturer _______________________

Name__________________________ Lecturer _______________________

1 (c) (10 pts) Find a general solution of

$$2x^2y'' - 3xy' + 2y = 0.$$

Solution: This d.e. is a Cauchy-Euler (or equidimensional) equation. The solution is of the form $y = x^m$. We substitute this into the d.e.

$$\begin{aligned} y &= x^m \\ y' &= mx^{m-1} \\ y'' &= m(m-1)x^{m-2} \\ [2m(m-1) - 3m + 2]x^m &= 0 \end{aligned}$$

So, the indicial equation is

$$\begin{aligned} 2m^2 - 5m + 2 &= 0 \\ (2m-1)(m-2) &= 0. \end{aligned}$$

The roots are $m = 2$, and $m = \frac{1}{2}$. The result is

$$\begin{aligned} y &= c_1x^2 + c_2x^{\frac{1}{2}} \\ &= c_1x^2 + c_2\sqrt{x}\,. \end{aligned}$$

2. (a) (12 pts) Find a general solution of

$$y'' - 5y' + 6y = 8e^t + 12t.$$

Solution: First, we solve the corresponding homogeneous equation by finding the roots of the characteristic polynomial.

$$\begin{aligned} r^2 - 5r + 6 &= 0 \\ (r-2)(r-3) &= 0 \\ r &= 2 \quad \text{and} \quad r = 3 \\ y_h &= c_1e^{2t} + c_2e^{3t} \end{aligned}$$

For

$$y'' - 5y' + 6y = 8e^t$$

we have (denoting the characteristic polynomial as $p(r)$)

$$\begin{aligned} y_{p1} &= \frac{8}{p(1)}e^t \\ &= \frac{8}{1^2 - 5 \cdot 1 + 6}e^t \\ &= \frac{8}{2}e^t \end{aligned}$$

Name__________________________ Lecturer _______________________

For

$$y'' - 5y' + 6y = 12t$$

we seek a soluton of the form

$$y_{p2} = At + B$$

Substituting, we have

$$\begin{aligned} -5A + 6(At + B) &= 12t \\ 6A &= 12 \\ -5A + 6B &= 0 \\ A &= 2 \\ B &= \frac{5}{3} \end{aligned}$$

Combining the three computations give the answer.

$$\begin{aligned} y &= y_h + y_{p1} + y_{p2} \\ &= c_1 e^{2t} + c_2 e^{3t} + 4e^t + 2t + \frac{5}{3}. \end{aligned}$$

2(b) (13 pts.) Find a general solution of

$$y'' + y = \sec x.$$

Solution. We will use variation of parameters. First, the solution to the corresponding homogeneous d.e. is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 \cos x + c_2 \sin x. \end{aligned}$$

We seek a solution of the form

$$y = v_1 y_1 + v_2 y_2.$$

The equations for v_1' and v_2' are as follows.

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= \cos x v_1' + \sin x v_2' = 0 \\ y_1' v_1' + y_2' v_2' &= -\sin x v_1' + \cos v_2' = \sec x \end{aligned}$$

Multiply the first equation by $(-\sin x)$, the second by $\cos x$ and add to obtain

$$\begin{aligned} \left(\sin^2 x + \cos^2 x\right) v_2' &= v_2' = 1 \\ v_2 &= x + c_2. \end{aligned}$$

Substitute into the first equation and obtain

$$\begin{aligned} \cos x v_1' + \sin x \cdot 1 &= 0 \\ v_1' &= \frac{-\sin x}{\cos x} = -\tan x \\ v_1 &= \ln(\cos x) + c_1 \\ y &= [c_1 + \ln(\cos x)] \cos x + [c_2 + x] \sin x. \end{aligned}$$

Name___________________________ Lecturer _______________________

3. (a) (10 pts.) Let

$$g(t) = \begin{cases} 1 \ \text{ for } 0 \le t \le 3 \\ e^{4t} \ \ \text{ for } 3 < t < \infty \end{cases}$$

Use the definition of the Laplace transform to find $\mathcal{L}\{g(t)\}$.
Solution:

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \int_0^\infty g(t)e^{-st}dt \\
&= \int_0^3 1 \cdot e^{-st}dt + \int_3^\infty e^{4t}e^{-st}dt \\
&= \int_0^3 1 \cdot e^{-st}dt + \lim_{L\to\infty} \int_3^L e^{(4-s)t}dt \\
&= \frac{e^{-st}}{-s}\bigg|_0^3 + \lim_{L\to\infty} \frac{e^{(4-s)t}}{4-s}\bigg|_3^L \\
&= \frac{e^{-3s}-1}{-s} + \lim_{L\to\infty} \frac{e^{(4-s)L} - e^{(4-s)3}}{4-s} \\
&= \frac{1-e^{-3s}}{s} + \frac{e^{(4-s)3}}{s-4} \qquad s > 4
\end{aligned}$$

.

Name__________________________ Lecturer _______________________

(b) (15 pts.) Solve using Laplace Transforms:

$$y'' - 4y' + 4y = te^t \qquad y(0) = 0,\ \ y'(0) = 1.$$

Solution: We take the Laplace Transform, using the following notation and formulae.

$$\begin{aligned} Y &= \mathcal{L}\{y\} \\ \mathcal{L}\left\{y'\right\} &= sY - y(0) \\ \mathcal{L}\left\{y''\right\} &= s^2Y - sy(0) - y'(0) \end{aligned}$$

The transform of the d.e. is

$$\begin{aligned} s^2Y - sy(0) - y'(0) - 4[sY - y(0)] + 4Y &= \frac{1}{(s-1)^2} \\ \left(s^2 - 4s + 4\right)Y - 1 &= \frac{1}{(s-1)^2} \\ \left(s^2 - 4s + 4\right)Y &= 1 + \frac{1}{(s-1)^2} \\ Y &= \frac{1}{(s-2)^2} + \frac{1}{(s-1)^2(s-2)^2} \end{aligned}$$

For the second fraction, we use the method of partial fractions to find an equivalent combination of fractions which are in the table of Laplace Transforms.

$$\frac{1}{(s-1)^2(s-2)^2} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-2)} + \frac{D}{(s-2)^2}$$

$$1 = A(s-1)(s-2)^2 + B(s-2)^2 + C(s-1)^2(s-2) + D(s-1)^2$$

Setting $s = 1$ and $s = 2$ yields two of the coefficients.

$$\begin{aligned} s = 1 \ &\Rightarrow\ 1 = B \\ s = 2 \ &\Rightarrow\ 1 = D \end{aligned}$$

The coefficient of s^3 gives a simple equation.

$$\begin{aligned} 0 &= A + C \\ C &= -A \end{aligned}$$

We obtain the constant term by setting $s = 0$ and then use the information found so far.

$$\begin{aligned} 1 &= -4A + 4B - 2C + D \\ 1 &= -4A + 4 + 2A + 1 \\ A &= 2 \end{aligned}$$

We combine everything and invert using the table.

$$\begin{aligned} Y &= \frac{1}{(s-2)^2} + \frac{1}{(s-1)^2(s-2)^2} \\ &= \frac{1}{(s-2)^2} + \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-2)} + \frac{D}{(s-2)^2} \\ &= \frac{1}{(s-2)^2} + \frac{2}{(s-1)} + \frac{1}{(s-1)^2} + \frac{-2}{(s-2)} + \frac{1}{(s-2)^2} \\ y &= te^{2t} + 2e^t + te^t - 2e^{2t} + te^{2t} \end{aligned}$$

Name___________________________ Lecturer ________________________

Name__________________________ Lecturer _______________________

4.) a.) (10 pts.) Use separation of variables, $u(x,t) = X(x)T(t)$, to find two ordinary differential equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$e^{x+t}\frac{\partial^2 u}{\partial x^2} - x^3(t+4)^5\frac{\partial^2 u}{\partial t^2} = 0.$$

Note: Do **not** solve these ordinary differential equations.
Solution:

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ \frac{\partial^2 u}{\partial x^2} &= X'' \cdot T \\ \frac{\partial^2 u}{\partial t^2} &= X \cdot T'' \\ e^{x+t}X''T - x^3(t+4)^5XT'' &= 0 \\ e^x e^t X''T &= x^3(t+4)^5XT'' \\ \frac{e^x X''}{x^3X} &= \frac{(t+4)^5T''}{e^tT} = \lambda \end{aligned}$$

The last step is the observation that one side is a function only of x and the other side is a function only of t so they must be constant. Taking one at a time produces the two O.D.E.s.
to match the

$$\begin{aligned} e^xX'' - \lambda x^3X &= 0 \\ (t+4)^5T'' - \lambda e^tT &= 0. \end{aligned}$$

b.) (15 pts.) Find

$$\mathcal{L}^{-1}\left\{\frac{4s^2-2s+30}{(s+2)\left(s^2-4s+13\right)}\right\}.$$

Solution: After completing the square in the quadratic factor in the denominator, we set up the partial fractions expansion needed.

$$\begin{aligned}
\frac{4s^2-2s+30}{(s+2)\left(s^2-4s+13\right)} &= \frac{4s^2-2s+30}{(s+2)\left(s^2-4s+4+9\right)} \\
&= \frac{4s^2-2s+30}{(s+2)\left[(s-2)^2+9\right]} \\
&= \frac{A}{s+2} + \frac{B(s-2)+C}{(s-2)^2+9}
\end{aligned}$$

The numerator of the second fraction could be $Bs+C$, but that would require some extra algebra to invert the Transform.

$$4s^2-2s+30 = A\left[(s-2)^2+9\right] + [B(s-2)+C](s+2)$$

Set $s = -2$.

$$\begin{aligned}
4 \cdot 4 - 2 \cdot (-2) + 30 &= A\left[4^2+3^2\right] \\
50 &= 25A \\
A &= 2
\end{aligned}$$

Next, set $s = 2$.

$$\begin{aligned}
4 \cdot 4 - 2 \cdot 2 + 30 &= 9A + 4C \\
42 &= 18 + 4C \\
C &= 6
\end{aligned}$$

Match the coefficients of s^2.

$$\begin{aligned}
4 &= A + B \\
B &= 2
\end{aligned}$$

Combine and invert.

$$\begin{aligned}
\frac{4s^2-2s+30}{(s+2)\left(s^2-4s+13\right)} &= \frac{A}{s+2} + \frac{B(s-2)+C}{(s-2)^2+9} \\
&= \frac{2}{s+2} + \frac{2(s-2)+6}{(s-2)^2+9}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{4s^2-2s+30}{(s+2)\left(s^2-4s+13\right)}\right\} &= \mathcal{L}^{-1}\left\{\frac{2}{s+2} + \frac{2(s-2)+6}{(s-2)^2+9}\right\} \\
&= 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+3^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+3^2}\right\} \\
&= 2e^{2t} + 2e^{2t}\cos 3t + 2e^{2t}\sin 3t
\end{aligned}$$

Name___________________________ Lecturer _______________________

5. (a) (15 pts.) Find the first five non-zero terms of the Fourier *sine* series for the function

$$f(x) = \begin{cases} \pi & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$

Solution:

$$f(x) = \sum_{k=1}^{\infty} \alpha_k \sin \frac{k\pi x}{L}$$

where

$$\alpha_k = \frac{2}{L}\int_0^L f(x) \sin \frac{k\pi x}{L} dx, \quad k = 1, 2, 3, \ldots$$

Here $L = 2\pi$ so

$$f(x) = \sum_{k=1}^{\infty} \alpha_k \sin\left(\frac{kx}{2}\right)$$

where

$$\alpha_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin\left(\frac{kx}{2}\right)dx, \quad k = 1, 2, 3, \ldots$$

Thus

$$\begin{aligned} \alpha_k &= \frac{1}{\pi}\int_0^{\pi} \pi \sin\left(\frac{kx}{2}\right)dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0 \sin\left(\frac{kx}{2}\right)dx = \\ &= \frac{2}{k}\left[-\cos\left(\frac{kx}{2}\right)\right]_0^{\pi} = -\frac{2}{k}\left[\cos\left(\frac{k\pi}{2}\right) - 1\right] \quad k = 1, 2, 3, \ldots \end{aligned}$$

Thus

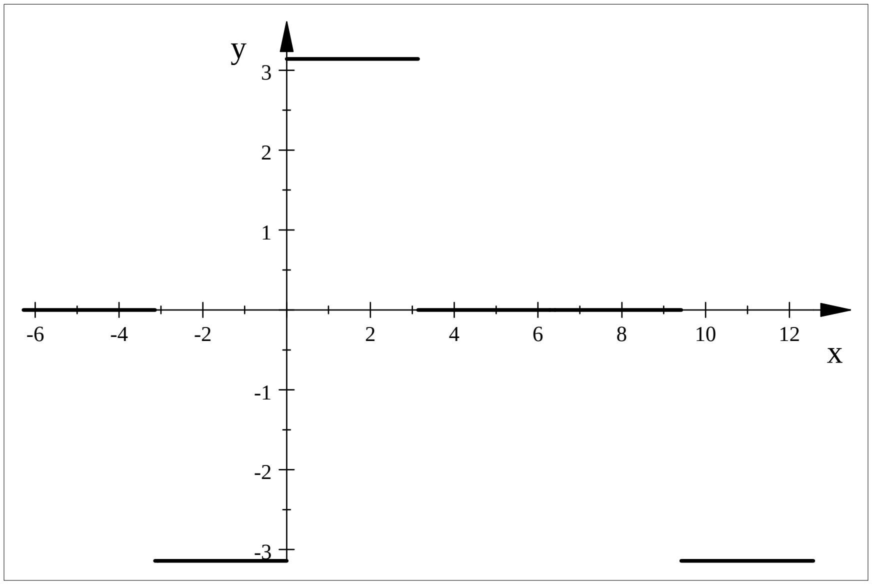
$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \alpha_k \sin\left(\frac{kx}{2}\right) = a_1 \sin\left(\frac{x}{2}\right) + a_2 \sin x + a_3 \sin\left(\frac{3x}{2}\right) + a_4 \sin 2x + a_5 \sin\left(\frac{5x}{2}\right) + a_6 \sin 3x + \cdots \\ &= 2\sin\left(\frac{x}{2}\right) + 2\sin x + \frac{2}{3}\sin\left(\frac{3x}{2}\right) + 0 \sin 2x + \frac{2}{5}\sin\left(\frac{5x}{2}\right) + \frac{2}{3}\sin 3x + \cdots \end{aligned}$$

5(b) (10 pts.) Sketch the graph of the function represented by the Fourier *sine* series in 5 (a) on $-2\pi < x < 4\pi$.

Solution:

π

Name___________________________ Lecturer ________________________



Name__________________________ Lecturer _______________________

6 (25 pts.) Solve

$$\text{PDE} \quad u_{xx} - 16u_{tt} = 0$$
$$\text{BCs} \quad u(0,t) = 0 \qquad u_x(1,t) = 0$$
$$\text{IC} \quad u(x,0) = -6\sin\left(\frac{3\pi x}{2}\right) + 13\sin\left(\frac{11\pi x}{2}\right)$$
$$\text{IC} \quad u_t(x,0) = 0$$

You must derive the solution. Your solution should not have any arbitrary constants in it. Show **all** steps.

Solution: This problem is from the Ma 221 Final Exam given in 10S.

Let $u(x,t) = X(x)T(t)$. Then the PDE implies

$$X''T = 16XT''$$

or

$$\frac{X''}{X} = 16\frac{T''}{T} = -\lambda^2$$

since we will need sines and cosines in the X part of the solution.
Thus

$$X'' + \lambda^2 X = 0$$
$$T'' + \frac{\lambda^2}{16}T = 0$$

The BCs are

$$X(0) = X'(1) = 0$$
$$X(x) = a_n \sin\lambda x + b_n \cos\lambda x$$

$X(0) = 0$ implies that $b_n = 0$, so

$$X(x) = a_n \sin\lambda x$$
$$X'(x) = a_n\lambda\cos\lambda x$$

so

$$X'(1) = a_n\lambda\cos\lambda = 0$$

Hence $\lambda = \frac{2n+1}{2}\pi,\ \ n = 0,1,2,\ldots$ and

$$X_n(x) = A_n\sin\left(\frac{2n+1}{2}\right)\pi x \quad n = 0,1,2,\ldots$$

Also

$$T'' + \frac{\lambda^2}{16}T = T'' + \frac{(2n+1)^2\pi^2}{64}T = 0$$
$$T_n(t) = c_n\sin\left(\frac{2n+1}{8}\right)\pi t + d_n\cos\left(\frac{2n+1}{8}\right)\pi t$$

$u_t(x,0) = 0$ implies that $c_n = 0$ and

$$T_n(t) = d_n\cos\left(\frac{2n+1}{8}\right)\pi t$$

Thus

$$u_n(x.t) = B_n \sin\left(\frac{2n+1}{2}\right)\pi x \cos\left(\frac{2n+1}{8}\right)\pi t \quad n = 0,1,2,\dots$$

Let

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x.t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2}\right)\pi x \cos\left(\frac{2n+1}{8}\right)\pi t$$

$$u(x,0) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2}\right)\pi x = -6\sin\left(\frac{3\pi x}{2}\right) + 13\sin\left(\frac{11\pi x}{2}\right)$$

Therefore $B_1 = -6, B_5 = 13$ and $B_n = 0$ for $n \neq 1,5$ so

$$u(x,t) = -6\sin\left(\frac{3\pi x}{2}\right)\cos\left(\frac{3\pi}{8}\right)t + 13\sin\left(\frac{11\pi x}{2}\right)\cos\left(\frac{11\pi}{8}\right)t$$

Name__________________________ Lecturer _______________________

7. (a) (13 pts.) Find a general solution of

$$y'' + y = x\cos x - \cos x$$

Solution: Note that $y_h = C_1 \cos x + C_2 \sin x$.

First we will find a particular solution for $\cos x$. Consider

$$y'' + y = -\cos x$$

and

$$v'' + v = -\sin x$$

Multiply the second equation by i and add it to the first equation.
Letting $w = y + iv$, we get

$$w'' + w = -(\cos x + i\sin x) = -e^{ix}$$

Since $p(\lambda) = \lambda^2 + 1$ and $p(i) = 0, p'(\lambda) = 2\lambda$, so $p'(i) = 2i \neq 0$

$$w_{p_1} = -\frac{xe^{ix}}{2i} = \frac{1}{2}ixe^{ix}$$

Hence

$$y_{p_1} = \operatorname{Re} w_{p_1} = -\frac{x}{2}\sin x$$

Now we shall find a particular solution for $x\cos x$. Consider

$$y'' + y = x\cos x$$

and

$$v'' + v = x\sin x$$

Multiplying the second equation by i, adding it to the first equation and letting $w = y + iv$, we have

$$w'' + w = x(\cos x + i\sin x) = xe^{ix}$$

Since e^{ix} is a homogeneous solution and xe^{ix} corresponds to a right hand side of e^{ix}, we let

$$w_{p_2} = \left(A_1 x + A_2 x^2\right)e^{ix}$$

to deal with a right side of the form xe^{ix}.

$$w'_{p_2} = (A_1 + 2A_2 x)e^{ix} + i\left(A_1 x + A_2 x^2\right)e^{ix}$$
$$w''_{p_2} = 2A_2 e^{ix} + 2i(A_1 + 2A_2 x)e^{ix} - \left(A_1 x + A_2 x^2\right)e^{ix}$$

Substituting into the DE leads to

$$2A_2 e^{ix} + 2i(A_1 + 2A_2 x)e^{ix} = xe^{ix}$$

Therefore

$$2A_2 + 2iA_1 = 0$$
$$4iA_2 = 1 \text{ or } A_2 = \frac{1}{4i} = -\frac{i}{4}$$

Then

$$A_1 = -\frac{1}{i}A_2 = \frac{1}{4}$$

$$w_{p_2} = \frac{1}{4}xe^{ix} - \frac{i}{4}x^2 e^{ix} = \left(\frac{1}{4}x - \frac{i}{4}x^2\right)(\cos x + i\sin x)$$

Name__________________________ Lecturer _______________________

$$y_{p_2} = \operatorname{Re} w_{p_2} = \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x$$

Thus

$$y = y_h + y_{p_1} + y_{p_2} = C_1\cos x + C_2\sin x - \frac{x}{2}\sin x + \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x$$

Another way to find the particular solution is shown below.

$$\begin{aligned}
y &= Ax\cos x + Bx^2\cos x + Cx\sin x + Dx^2\sin x \\
y' &= A\cos x - Ax\sin x + 2Bx\cos x - Bx^2\sin x + C\sin x + Cx\cos x + 2Dx\sin x + Dx^2\cos x \\
y'' &= -A\sin x - A\sin x - Ax\cos x + 2B\cos x - 2Bx\sin x - 2Bx\sin x \\
&\quad - Bx^2\cos x + C\cos x + C\cos x - Cx\sin x + 2D\sin x + 2Dx\cos x + 2Dx\cos x - Dx^2\sin x \\
y'' + y &= -A\sin x - A\sin x + 2B\cos x - 2Bx\sin x \\
&\quad - 2Bx\sin x + C\cos x + C\cos x + 2D\sin x + 2Dx\cos x + 2Dx\cos x \\
&= x\cos x - \cos x
\end{aligned}$$

Next, we match coefficients of the four functions in the equation.

$$\begin{aligned}
\sin x &: \quad -2A + 2D = 0 \\
x\sin x &: \quad -4B = 0 \\
\cos x &: \quad 2B + 2C = -1 \\
x\cos x &: \quad 4D = 1
\end{aligned}$$

So, as before,

$$A = D = \frac{1}{4}$$

$$B = 0$$

$$C = -\frac{1}{2}$$

$$y_p = \frac{1}{4}x\cos x - \frac{1}{2}x\sin x + \frac{1}{4}x^2\sin x.$$

$$y = y_h + y_p = C_1\cos x + C_2\sin x - \frac{x}{2}\sin x + \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x$$

Name___________________________ Lecturer _______________________

7 (b) (12 pts.) Find the power series solution to

$$y'' - xy = 0$$

near $x = 0$. Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first six nonzero terms of the solution.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

so

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

.

The differential equation ⇒

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Let $k + 1 = n - 2$ in the first series. That is, $n = k + 3$. Then we have

$$\sum_{k=-1}^{\infty} a_{k+3}(k+3)(k+2)x^{k+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

We replace k and n by m and have

$$(2)(1)a_2 + \sum_{m=0}^{\infty} [a_{m+3}(m+3)(m+2) - a_m] x^{m+1} = 0$$

Thus $a_2 = 0$ and

$$a_{m+3}(m+3)(m+2) - a_m = 0 \ \text{ for } m = 0, 1, 2, \ldots$$

This or

$$a_{m+3} = \frac{a_m}{(m+3)(m+2)} \quad \text{for } m = 0, 1, 2, \ldots$$

is the recurrence relation. Hence we have

$$a_3 = \frac{1}{3(2)} a_0$$

$$a_4 = \frac{1}{4(3)} a_1$$

$$a_5 = 0$$

$$a_6 = \frac{1}{6(5)} a_3 = \frac{1}{6(5)(3)(2)} a_0$$

$$a_7 = \frac{1}{(7)(6)} a_4 = \frac{1}{(7)(6)4(3)} a_1$$

Thus

Name___________________________ Lecturer _______________________

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0\left[1 + \frac{1}{3(2)}x^3 + \frac{1}{6(5)(3)(2)}x^6 + \cdots\right] + a_1\left[x + \frac{1}{4(3)}x^4 + \frac{1}{(7)(6)4(3)}x^7 + \cdots\right]$$

SNB check: $y'' - xy = 0$, Series solution is:
$\left\{y(0) + xy'(0) + \frac{1}{6}x^3 y(0) + \frac{1}{12}x^4 y'(0) + \frac{1}{180}x^6 y(0) + O(x^7)\right\}$

Name___________________________ Lecturer _______________________

8 (a) (15 pts.) Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0 \qquad y'(0) = y'(2\pi) = 0$$

Be sure to consider the cases $\lambda < 0, \lambda = 0,$ and $\lambda > 0.$

Solution: There are three cases to deal with, $\lambda > 0, \lambda = 0, \lambda < 0.$

I. $\lambda < 0.$ Let $\lambda = -\alpha^2,$ where $\alpha \neq 0.$ Then the DE is

$$y'' - \alpha^2 y = 0$$

and

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

Hence

$$y' = c_1 \alpha e^{\alpha x} - c_{2\alpha} e^{-\alpha x}$$

The initial conditions imply

$$y'(0) = c_1\alpha - c_2\alpha = 0$$

$$y'(2\pi) = c_1 \alpha e^{2\alpha\pi} - c_2 \alpha e^{-2\alpha\pi} = 0$$

The first equation implies that $c_1 = c_2$ so the second equation implies that

$$c_1\alpha\left(e^{2\alpha\pi} - e^{-2\alpha\pi}\right) = 0$$

Since $\alpha \neq 0$ and $e^{\alpha\pi} - e^{-\alpha\pi} \neq 0,$ then $c_1 = c_2 = 0$ and there are no eigenvalues for $\lambda < 0.$

II. $\lambda = 0.$ The DE now is $y'' = 0$ so $y = c_1 x + c_2$ and $y'(x) = c_1.$ Thus the initial conditions imply $c_1 = 0$ and $y(x) = b_0$ where b_0 is any nonzero constant is a solution.

III.$\lambda > 0.$ Let $\lambda = \beta^2,$ where $\beta \neq 0.$ The DE becomes $y'' + \beta^2 y = 0$ and

$$y(x) = a\sin\beta x + b\cos\beta x$$

Thus

$$y'(x) = a\beta\cos\beta x - b\beta\sin\beta x$$

$y'(0) = a\beta = 0$ so $a = 0.$

$$y'(2\pi) = -b\beta\sin 2\beta\pi = 0$$

Thus

$$2\beta\pi = n\pi$$

and

$$\beta = \frac{n}{2} \quad n = 1,2,\ldots$$

The eigenvalues are

$$\lambda = \beta^2 = \left(\frac{n}{2}\right)^2 \quad n = 1,2,\ldots$$

and the eigenfunctions are

$$y_n = b_n\cos\left(\frac{n}{2}\right)x \quad n = 1,2,\ldots$$

Thus have the eigenvalues and eigenfunctions

$$y_n = b_n\cos\left(\frac{n}{2}\right)x \quad n = 0,1,2,\ldots$$

Name__________________________ Lecturer _______________________

Name__________________________ Lecturer _______________________

8(b) (10 pts.) Solve

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{xy^2} \quad x > 0$$

Solution: This is a Bernoulli equation and we may rewrite it as

$$y^2 y' + \frac{y^3}{x} = \frac{1}{x}$$

Let $z = y^3$ so $z' = 3y^2 y'$ and the above DE becomes

$$\frac{1}{3} z' + \frac{z}{x} = \frac{1}{x}$$

or

$$z' + \frac{3}{x} z = \frac{3}{x}$$

The integrating factor is $e^{\int \frac{3}{x} dx} = x^3$. Multiplying the DE by this we have

$$x^3 z' + 3x^2 z = 3x^2$$

or

$$\frac{d\left(x^3 z\right)}{dx} = 3x^2$$

so

$$x^3 z = x^3 + c$$

or

$$y^3 = z = 1 + \frac{c}{x^3}$$

so

$$y = \left(1 + \frac{c}{x^3}\right)^{\frac{1}{3}}$$

Name___________________________ Lecturer ________________________

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$		
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^n}$	$n \geq 1$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$		$s > a$
$\sin bt$	$\dfrac{b}{s^2 + b^2}$		$s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$		$s > 0$
$e^{at}f(t)$	$\mathcal{L}\{f\}(s-a)$		
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f\}(s))$		

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int x \cos bx dx = \frac{1}{b^2}(\cos bx + bx \sin bx) + C$
$\int x \sin bx dx = \frac{1}{b^2}(\sin bx - bx \cos bx) + C$
$\int \left(\frac{e^{-t}}{1+e^{-t}}\right) dt = -\ln(1 + e^{-t}) + C$
$\int \left(\frac{e^{-2t}}{1+e^{-t}}\right) dt = \ln(1 + e^{-t}) - e^{-t} + C$
$\int x e^{ax} dx = \frac{1}{a^2}(axe^{ax} - e^{ax}) + C$