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MA221 FINAL EXAM Solutions May 2000

You may not use a calculator on this exam.

I pledge my honor that I have abided by the Stevens Honor System. ____________________________

Part I: **Answer all questions**

1. Find the general solution for the following differential equations

(a) (8 pts)

$$y' + \frac{4}{x}y = x^4$$

Solution:

The integrating factor is

$$e^{\int P dx} = e^{\int \frac{4}{x} dx} = x^4$$

Multiplying the DE by x^4 we get

$$x^4 y' + 4x^3 y = x^8$$

or

$$\left(x^4 y\right)' = x^8$$

Thus

$$x^4 y = \frac{x^9}{9} + C$$

So the solution is

$$y(x) = \frac{1}{9}x^5 + \frac{1}{x^4}C$$

(b) (9 pts)

$$y'' + 4y' + 4y = 0$$

Solution:

The characteristic equation is

$$r^4 + 4r + 4 = 0$$

or

$$(r+2)^2 = 0$$

Thus –2 is a repeated root so the exact solution is:

$$y(x) = C_1 e^{-2x} + C_2 e^{-2x} x$$

(c) (8 pts)

$$(y^2 - xy)dx + x^2 dy = 0$$

Solution:

We rewrite the equation as

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$$x^2\frac{dy}{dx} - xy = -y^2$$

or

$$y' - \frac{1}{x}y = -\frac{y^2}{x^2}$$

This is a Bernoulli equation. to solve it we multiply both sides of the above equation by y^{-2} and get

$$y'y^{-2} - \frac{1}{x}y^{-1} = -\frac{1}{x^2}$$

Let $z = y^{-1}$. Then $z' = -y^{-2}y'$. The De can now be written as

$$z' + \frac{1}{x}z = \frac{1}{x^2}$$

We multiply the DE by $e^{\int \frac{1}{x}dx} = x$ and get

$$(xz)' = \frac{1}{x}$$

Thus

$$xz = \frac{x}{y} = \ln x + C_1$$

and the exact solution is:

$$y(x) = \frac{x}{\ln x + C_1}$$

2. (a) (12 pts) Find the general solution of

$$y'' + 4y = x^2 - 1 + 2e^{-2x}.$$

Solution:

$$p(r) = r^2 + 4. \qquad p(r) = 0 \Rightarrow r = \pm 2i.$$

$$y_h = C_1 \sin 2x + C_2 \cos 2x$$

$$y_{p_1} = Ax^2 + Bx + C$$

$$y'_{p_1} = 2Ax + B, \quad y''_p = 2A$$

$$\Rightarrow 2A + 4\left(Ax^2 + Bx + C\right) = x^2 - 1.$$

$$4Ax^2 = x^2 \Rightarrow A = \frac{1}{4}. \quad B = 0$$

$$2A + 4C = \frac{1}{2} + 4C = -1 \Rightarrow C = -\frac{3}{8}$$

Thus $y_{p_1} = \frac{1}{4}x^2 - \frac{3}{8}$

$$y_{p_2} = \frac{2e^{-2x}}{p(-2)} = \frac{1}{4}e^{-2x}$$

$\Rightarrow$

$$y_g = C_1 \sin 2x + C_2 \cos 2x + \frac{1}{4}x^2 - \frac{3}{8} + \frac{1}{4}e^{-2x}$$

2(b) (13 pts) Given that $y_1 = x$ and $y_2 = x^4$ are linearly independent solutions for

$$x^2y'' - 4xy' + 4y = 0,$$

use variation of parameters to find a particular solution of

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$$y'' - \frac{4}{x}y' + \frac{4}{x^2}y = x^2 + x$$

, Exact solution is: $y(x) = \frac{1}{18}x^3(-2x - 9 + 6(\ln x)x) + C_1 x + C_2 x^4$

Solution:

Cauchy-Euler equation, so characteristic equation is

$m^2 + (-4-1)m + 4 = 0 \Rightarrow m^2 - 5m + 4 = 0 \Rightarrow m = 1, 4.$

$\Rightarrow y_h = C_1 x + C_2 x^4$. Use Variation of Parameters to find y_p.

$\Rightarrow y_p = v_1 y_1 + v_2 y_2 = v_1 x + v_2 x^4.$

$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = x(4x^3) - x^4(1) = 3x^4.$

$$v_1 = -\int \frac{(x^4)\left(x^2+x\right)}{\left(3x^4\right)}dx = -\frac{1}{3}\int\left(x^2+x\right)dx = -\frac{1}{9}x^3 - \frac{1}{6}x^2.$$

$$v_2 = \int \frac{(x)(x^2+x)}{(3x^4)}dx = \frac{1}{3}\int\left(\frac{1}{x} + \frac{1}{x^2}\right)dx = \frac{1}{3}\ln x - \frac{1}{3x}.$$

$$\Rightarrow y_p = \left(-\frac{1}{9}x^3 - \frac{1}{6}x^2\right)x + \left(\frac{1}{3}\ln x - \frac{1}{3x}\right)x^4 = -\frac{1}{9}x^4 - \frac{1}{6}x^3 + \frac{1}{3}x^4\ln x - \frac{1}{3}x^3$$

$$\Rightarrow y_g = C_1 x + C_2 x^4 - \frac{1}{9}x^4 - \frac{1}{2}x^3 + \frac{1}{3}x^4\ln x.$$

3. (a) (10 pts) Find

$$\mathcal{L}^{-1}\left(\frac{s+4}{s^2+4s+8}\right)$$

Solution:

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s+4}{s^2+4s+8}\right) &= \mathcal{L}^{-1}\left(\frac{s+4}{(s+2)^2+4}\right)\\ &= \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+4} + \frac{2}{(s+2)^2+4}\right)\\ &= e^{-2t}\cos 2t + e^{-2t}\sin 2t\end{aligned}$$

(b) (15 pts) Use Laplace Transforms to solve:

$$y'' - 3y' + 2y = e^{-x}, \quad y(0) = 1, \ y'(0) = 1$$

Solution:

$$\mathcal{L}\left(y'' - 3y' + 2y\right) = \mathcal{L}(e^{-x}) = \frac{1}{s+1}$$

Thus

$$\left(s^2 - 3s + 2\right)\mathcal{L}(y) - sy(0) - y'(0) + 3y(0) = \frac{1}{s+1}$$

$$\left(s^2 - 3s + 2\right)\mathcal{L}(y) = \frac{1}{s+1} + s - 2$$

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$$\begin{aligned}\mathcal{L}(y) &= \frac{1}{(s+1)(s-1)(s-2)} + \frac{s}{(s-1)(s-2)} - \frac{2}{(s-1)(s-2)} \\ &= \frac{-1+s^2-s}{(s+1)(s-1)(s-2)} \\ &= \frac{1}{6(s+1)} + \frac{1}{2(s-1)} + \frac{1}{3(s-2)}\end{aligned}$$

Thus

$$y(x) = \frac{1}{2}e^{x} + \frac{1}{3}e^{2x} + \frac{1}{6}e^{-x}$$

4. (a) (12 pts) Find the eigenvalues and eigenfunctions for the problem

$$y'' + \lambda y = 0; \qquad y(0) = 0; \qquad y'(\pi) = 0.$$

Be sure to consider the cases $\lambda > 0, \lambda = 0.$ and $\lambda < 0.$

Solution:

Case I: $\lambda < 0.$ Let $\lambda = -k^2.$ Then $r^2 - k^2 = 0 \Rightarrow r = \pm k.$

So $y = C_1 e^{kx} + C_2 e^{-kx}.$ $y' = C_1 k e^{kx} - C_2 k e^{-kx}$

$y(0) = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2.$

$y'(\pi) = C_1 k e^{k\pi} + C_1 k e^{-k\pi} = 0 \Rightarrow C_1 = 0 \Rightarrow C_2 = 0 \Rightarrow y = 0.$

Case II: $\lambda = 0.$ $y = C_1 x + C_2.$ $y' = C_1 \Rightarrow y'(\pi) = C_1 = 0.$

$y(0) = 0$ implies that $C_2 = 0$, Thus $y = 0,$ and 0 is not an eigenvalue.

Case III: $\lambda = k^2 > 0.$ Then $y = C_1 \sin kx + C_2 \cos kx.$

$y'(x) = C_1 k \cos kx - C_2 k \sin kx.$

$y(0) = C_2 k = 0 \Rightarrow C_2 = 0.$

$y'(\pi) = C_1 k \cos k\pi = 0 \Rightarrow$ if $C_1 \neq 0,$ $k\pi = (2n+1) * \frac{\pi}{2}$

$\Rightarrow k = \frac{2n+1}{2},$ $n = 0, 1, 2, \cdots.$

Therefore $y_n(x) = C_n \sin\left(\frac{2n+1}{2}x\right),$ $\lambda_n = \frac{(2n+1)^2}{4},$ $n = 0, 1, 2, \cdots$

4(b) (13 pts) Find the power series solution to

$$y'' - xy' + y = 0$$

near $x = 0.$ Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first four nonzero terms the solution.

Solution:

Let $y = \sum_{n=0}^{\infty} a_n x^n.$ $y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$ $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$

Plugging in gives

$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$

Let $k = n - 2$

$n = k + 2$

$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$

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$$\Rightarrow (2)(1)a_2 + a_0 + (3)(2)a_3 x + \sum_{k=2}^{\infty}[(k+2)(k+1)a_{k+2} - (k-1)a_k]x^k = 0.$$

$$\Rightarrow a_2 = -\frac{a_0}{2}, \qquad a_3 = 0 \text{ and the Recurrence relation: } a_{k+2} = \frac{a_k(k-1)}{(k+1)(k+2)}, \qquad k = 2,3,\ldots$$

$$k = 2 : a_4 = \frac{a_2}{3(4)} = -\frac{a_0}{24}. \qquad k = 3 : a_5 = 0.$$

$$\Rightarrow y = a_0\left(1 - \frac{x^2}{2} - \frac{x^4}{24} + \cdots\right) + a_1 x.$$

5. (a) (15 pts) Solve

$$y'' + y = \tan x$$

Solution: See Example 1 Page 215 of text.
The homogeneous solutions are $\cos x$ and $\sin x$. Thus

$$y_p = v_1(x)\cos x + v_2(x)\sin x$$

The two equations for v_1' and v_2' are

$$(\cos x)v_1' + (\sin x)v_2' = 0$$
$$(-\sin x)v_1' + (\cos x)v_2' = \tan x$$

Solving yields

$$v_1' = -\tan x \sin x$$
$$v_2' = \tan x \cos x = \sin x$$

Thus

$$v_1 = -\int \tan x \sin x dx = \sin x - \ln|\sec x + \tan x|$$
$$v_2 = \int \sin x dx = -\cos x$$

Thus

$$y_p = -(\cos x)\ln|\sec x + \tan x|$$

Finally

$$y = c_1 \cos x + c_2 \sin x - (\cos x)\ln|\sec x + \tan x|$$

(b) (10 pts) Compute the Wronskian for the solutions to the differential equation

$$y'' - 2y' + y = 0$$

Solution:
$y'' - 2y' + y = 0$, Exact solution is: $y(x) = C_1 e^x + C_2 e^x x$. Let $y_1 = e^x$ and $y_2 = xe^x$

$$W[y_1, y_2] = \det\begin{bmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{bmatrix} = e^{2x}$$

6. (a) (10 pts) Use separation of variables, $u(x,t) = X(x)T(t)$, to find two ordinary differential

equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$-3x^2t^4\frac{\partial^2 u}{\partial x^2}+(x-2)^4(t+6)^3\frac{\partial^2 u}{\partial t^2}=0.$$

Note: Do **not** solve these ordinary differential equations.
Solution:

$$-3x^2t^4X''(x)T(t)+(x-2)^4(t+6)^3X(x)T''(t)=0$$

$$\Rightarrow \frac{-3x^2X''}{(x-2)^4X}=-\frac{(t+6)^3T''}{t^4T}=k.$$

$$\Rightarrow 3x^2X''+k(x-2)^4X=0 \text{ and } (t+6)^4T''+kt^4T=0.$$

6(b) (15 pts) Use the method of separation of variables to solve

$$u_{xx}=u_t$$
$$u(0,t)=0 \qquad u_x(1,t)=0$$
$$u(x,0)=-3\sin\frac{\pi}{2}x+12\sin\frac{9\pi}{2}x$$

You must derive the solution. Your solution should not have any arbitrary constants in it.
SOLUTION:

$u(x,t)=X(x)T(t) \Rightarrow X''(x)T(t)=X(x)T'(t) \Rightarrow \frac{X''}{X}=\frac{T'}{T}=K$

$$X''-KX=0$$
$$T'-KT=0$$

Now apply the BC's.
$u(0,t)=X(0)T(t)=0$
$u_x(1,t)=X'(1)T(t)=0$
We do not want the problem to be trivial, hence we shall assume $T(t)\neq 0$ and hereby shall obtain the two BC's for $X(x)$

$$X(0)=0$$
$$X'(1)=0$$

Consider the three cases for the values of K in $X''-KX=0$ with the above BC's.

Case 1. $K=0$. $r^2=0$; $r_{1,2}=0 \Rightarrow y=C_1+C_2x$
$X(0)=C_1=0$; $X'(1)=C_2=0$, i.e., a trivial case.

Case 2. $K>0$. $K=b^2$.
$r^2=b^2$, $r_{1,2}=\pm b$.
$X(x)=C_1e^{bx}+C_2e^{-bx}$; $X'=bC_1e^{bx}-bC_2e^{-bx}$
$X(0)=C_1+C_2=0 \Rightarrow C_1=-C_2$

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$X'(1) = bC_1e^b + bC_1e^{-b} = 0 = bC_1(e^b - e^{-b})$ => $C_1 = 0 = C_2$, i.e., a trivial case again.

Case 3. $K < 0,\ K = -\beta^2$.

$r_{1,2} = \pm\beta i$

$X(x) = C_1\cos\beta x + C_2\sin\beta x$

$X(0) = 0 = C_1\cos 0;\quad C_1 = 0$ => $X(x) = C_2\sin\beta x$.

$X'(1) = C_2\beta\cos\beta = 0,\ \beta = (2n+1)\frac{\pi}{2}$ =>

$$K = -\beta^2 = -(2n+1)^2\frac{\pi^2}{4}$$

$$X_n(x) = a_n\sin\left[(2n+1)\frac{\pi}{2}x\right]$$

$n = 0, 1, 2, \ldots$

Now $T' - KT = 0$ <=> $T' + (2n+1)^2\frac{\pi^2}{4}T = 0$ => after solving the 1st order DE

$$T_n(t) = C_ne^{-\frac{\pi^2}{4}t(2n+1)^2}$$

$u(x,t) = X(x)T(t)$

$$u_n(x,t) = a_n\sin\left[(2n+1)\frac{\pi}{2}x\right]\left[C_ne^{-\frac{\pi^2}{4}t(2n+1)^2}\right]$$

We include C_n into a_n, obtaining b_n.

$$u_n(x,t) = b_ne^{-\frac{\pi^2}{4}t(2n+1)^2}\sin\left[(2n+1)\frac{\pi}{2}x\right]$$

Thus

$$u(x,t) = \sum_{n=0}^{\infty}u_n(x,t) = \sum_{n=0}^{\infty}b_ne^{-\frac{\pi^2}{4}t(2n+1)^2}\sin\left[(2n+1)\frac{\pi}{2}x\right]$$

Now, applying the IC,

$$u(x,0) = \sum_{n=0}^{\infty}b_n\sin\left[(2n+1)\frac{\pi}{2}x\right].$$

>From the IC => $u(x,0) = b_1\sin\frac{\pi}{2}x + b_4\sin\frac{9\pi}{2}x$, i.e., $b_0 = -3$ and $b_4 = 12$. All other b_i's are equal to zero.

- Finally,

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$$u(x,t) = -3\sin\left(\frac{\pi}{2}x\right)e^{-\frac{\pi^2}{4}t} + 12\sin\left(9\frac{\pi}{2}x\right)e^{-\frac{81\pi^2}{4}t}$$

Part II: **Choose any two (2) questions**.

7. (a) (13 pts) Consider the function

$$f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$$

Write down the expression for the Fourier cosine series for this function. Give a formula for the coefficients in this expression with appropriate limits. DO NOT EVALUATE THE COEFFICIENTS.

Solution: $L = \pi$

$$f(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos nx$$

$$b_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} xdx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)dx\right]$$

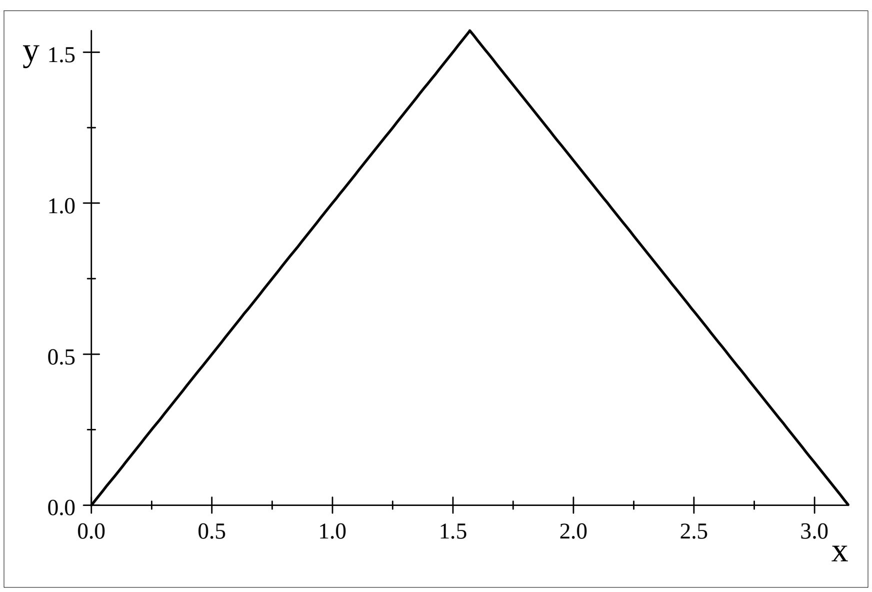
$$b_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\cos nxdx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\cos nxdx\right]$$

(b) (12 pts) Sketch the graph of the function represented by the Fourier cosine series in 7 (a) on $-\pi \le x \le 3\pi$.

$\frac{\pi}{2} = 1.5708, \pi = 3.1416, \frac{3\pi}{2} = 4.7124, 2\pi = 6.2832, \frac{5\pi}{2} = 7.8540, 3\pi = 9.4248$

We first graph the funtion on $[0,\pi]$. The graph is shown below.

x



Since this is a Fourier cosine expansion, the function is extended to $[-\pi, 0]$ by simply reflecting it

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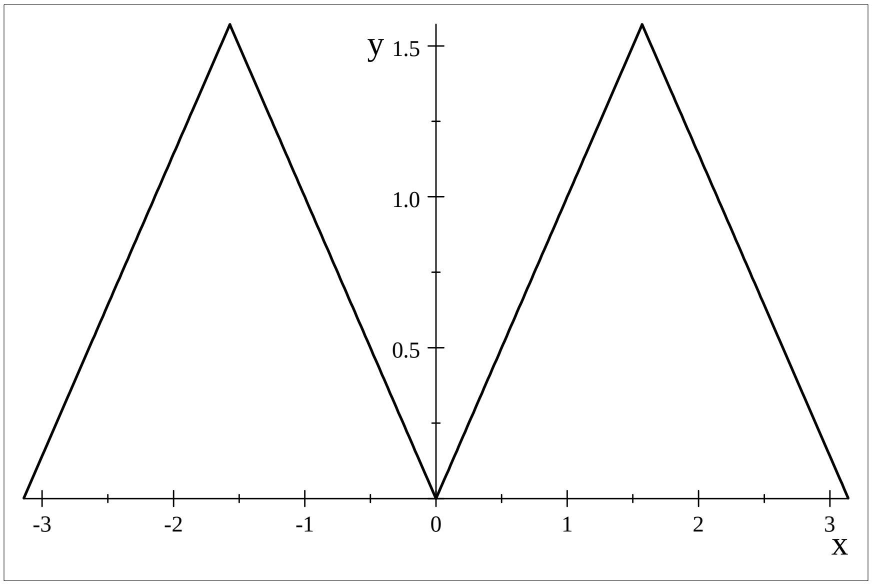
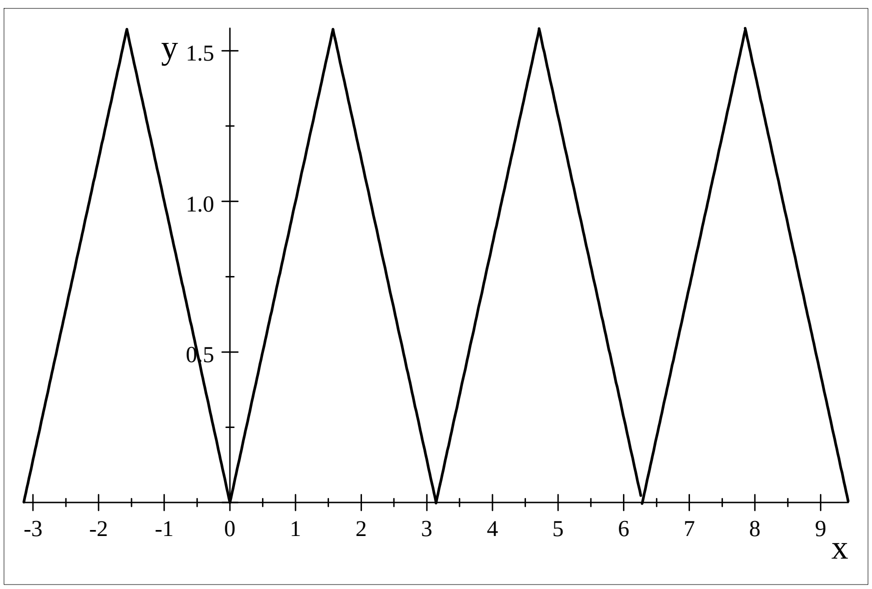
across the y –axis. Thus the graph on $[-\pi,\pi]$ is
$-x$



Since we now have the graph on $[-\pi,\pi]$ and the graph has period 2π, we simply "pick up" the graph and move it to the right to get what is looks like on $[\pi,3\pi]$. Thus



8. (25 pts) Find the solution to the problem

$$\begin{aligned} u_{tt} &= 100u_{xx} \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= x(\pi-x) \\ u_t(x,0) &= 0 \end{aligned}$$

Your answer must display *all* coefficients.

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Note:

$$\int u^2 \sin au du = \frac{1}{a^3}\left(-a^2u^2 \cos au + 2\cos au + 2au \sin au\right) + C$$
$$\int u \sin au du = \frac{1}{a^2}(\sin au - au \cos au) + C$$

Solution:
Assume

$$u(x,t) = X(x)T(t)$$

The the PDE implies

$$\frac{X''}{X} = \frac{T''}{100T} = k$$

Since the left hand side is a function of *x* only, and the right hand side is a function of *t* only. Thus we have two ODEs

$$X'' - kX = 0$$
$$T'' - 100kT = 0$$

The BCs imply

$$X(0) = X(\pi) = 0$$

and

$$T'(0) = 0$$

The solution to the BVP for *X* is

$$X_n(x) = c_n \sin n\pi x$$

with

$$k = -n^2 \quad n = 1,2,\ldots$$

The solution to the equation for *T* that satisfies the initial condition $T'(0) = 0$ is

$$T_n(t) = d_n \cos 10nt$$

Hence

$$u_n(x,t) = A_n \sin nx \cos 10nt \quad n = 1,2,\ldots$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \cos 10nt$$

The last condition to be satisfied is

$$u(x,0) = x(\pi - x) = \sum_{n=1}^{\infty} A_n \sin nx$$

Thus the $A_n's$ are the Fourier sine coefficients of this expansion and

$$A_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin nx dx$$
$$= -\left(\frac{2}{\pi}\right)\left(\frac{2\cos\pi n - 2}{n^3}\right)$$
$$= \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{8}{\pi n^3} & \text{for } n \text{ odd} \end{cases}$$

where the given formulas have been used to evaluate the integrals. Thus

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos 10nt(\sin nx)$$

where

$$A_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{8}{\pi n^3} & \text{for } n \text{ odd} \end{cases}$$

9. (a) (12 pts) Find the Laplace transform of

$$f(x) = x\cos\sqrt{7}x$$

Solution: From the table

$\mathcal{L}(xf(x)) = -\frac{d}{ds}\hat{f}(s)$

$\mathcal{L}\left(\cos\left(\sqrt{7}t\right)\right) = \frac{s}{s^2+7}$

$\frac{d}{ds}\left(\frac{s}{s^2+7}\right) = -\frac{s^2-7}{(s^2+7)^2}$

Thus $\mathcal{L}\left(x\cos\sqrt{7}x\right) = \frac{s^2-7}{(s^2+7)^2}$

(b) (12 pts) Show that the equation

$$\left(\frac{y^2}{2} + 2ye^x\right)dx + (y + e^x)dy = 0$$

is *not* exact and then find an integrating factor for this equation. Do *not* solve the equation.

Solution:

$\frac{y^2}{2} + 2ye^x + (y + e^x)y' = 0$

$\Rightarrow \quad \left(\frac{y^2}{2} + 2ye^x\right)dx + (y + e^x)dy = 0$

$M_y = y + 2e^x \qquad N_x = e^x \qquad$ therefore not exact.

$u\left(\frac{y^2}{2} + 2ye^x\right)dx + u(y + e^x)dy = 0$

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$$u_y\left(\frac{y^2}{2} + 2ye^x\right)dx + u(y + e^x)dy = u_x(y + e^x) + u(e^x)$$

$$u_y = 0 \qquad \Rightarrow \qquad u(y + e^x) = u_x(y + e^x)$$

$$u = u_x \qquad \Rightarrow \qquad u = e^x$$

$$\left(e^x\frac{y^2}{2} + 2ye^{2x}\right)dx + \left(ye^x + e^{2x}\right)dy = 0$$

$$M_y = ye^x + 2e^{2x}$$

$$N_x = ye^x + 2e^{2x}$$

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Table of Laplace Transforms

$f(t)$	$\hat{f}(s)$	
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	$n \geq 1 \quad s > 0$
$\sin ax$	$\frac{a}{s^2 + a^2}$	$s > a$
$\cos ax$	$\frac{s}{s^2 + a^2}$	$s > a$
$e^{-bt}f(t)$	$\hat{f}(s+b)$	
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} \hat{f}(s)$	