MA221 FINAL EXAM Solutions May 2000

You may not use a calculator on this exam.
I pledge my honor that I have abided by the Stevens Honor System.

Part I: Answer all questions
1. Find the general solution for the following differential equations
(a) (8 pts)

\[ y' + \frac{4}{x}y = x^4 \]

Solution:
The integrating factor is

\[ e^\int Pdx = e^{\int \frac{4}{x}dx} = x^4 \]

Multiplying the DE by \( x^4 \) we get

\[ x^4y' + 4x^3y = x^8 \]

or

\[ (x^4y)' = x^8 \]

Thus

\[ x^4y = \frac{x^9}{9} + C \]

So the solution is

\[ y(x) = \frac{1}{9}x^5 + \frac{1}{x^4}C \]

(b) (9 pts)

\[ y'' + 4y' + 4y = 0 \]

Solution:
The characteristic equation is

\[ r^4 + 4r + 4 = 0 \]

or

\[ (r + 2)^2 = 0 \]

Thus \(-2\) is a repeated root so the exact solution is:

\[ y(x) = C_1e^{-2x} + C_2e^{-2x}x \]

(c) (8 pts)

\[ (y^2 - xy)dx + x^2dy = 0 \]

Solution:
We rewrite the equation as
\[ x^2 \frac{dy}{dx} - xy = -y^2 \]

or

\[ y' - \frac{1}{x}y = -\frac{y^2}{x^2} \]

This is a Bernoulli equation. To solve it, we multiply both sides of the above equation by \( y^{-2} \) and get

\[ y'y^{-2} - \frac{1}{x}y^{-1} = -\frac{1}{x^2} \]

Let \( z = y^{-1} \). Then \( z' = -y^{-2}y' \). The DE can now be written as

\[ z' + \frac{1}{x}z = \frac{1}{x^2} \]

We multiply the DE by \( e^{\int \frac{1}{x}dx} = x \) and get

\[ (xz)' = \frac{1}{x} \]

Thus

\[ xz = \frac{x}{y} = \ln x + C_1 \]

and the exact solution is:

\[ y(x) = \frac{x}{\ln x + C_1} \]

2. (a) (12 pts) Find the general solution of

\[ y'' + 4y = x^2 - 1 + 2e^{-2x}. \]

Solution:

\[ p(r) = r^2 + 4. \quad p(r) = 0 \Rightarrow r = \pm 2i. \]

\[ y_h = C_1 \sin 2x + C_2 \cos 2x \]

\[ y_{p_1} = Ax^2 + Bx + C \]

\[ y_{p_1}' = 2Ax + B, \quad y_{p_1}'' = 2A \]

\[ \Rightarrow 2A + 4(2Ax^2 + Bx + C) = x^2 - 1. \]

\[ 4Ax^2 = x^2 \Rightarrow A = \frac{1}{4}, \quad B = 0 \]

\[ 2A + 4C = \frac{1}{2} + 4C = -1 \Rightarrow C = -\frac{3}{8} \]

Thus

\[ y_{p_1} = \frac{1}{4}x^2 - \frac{3}{8} \]

\[ y_{p_2} = \frac{2e^{-2x}}{p(-2)} = \frac{1}{4}e^{-2x} \]

\[ \Rightarrow \quad y_g = C_1 \sin 2x + C_2 \cos 2x + \frac{1}{4}x^2 - \frac{3}{8} + \frac{1}{4}e^{-2x} \]

2(b) (13 pts) Given that \( y_1 = x \) and \( y_2 = x^4 \) are linearly independent solutions for

\[ x^2y'' - 4xy' + 4y = 0, \]

use variation of parameters to find a particular solution of
\[ y'' - \frac{4}{x}y' + \frac{4}{x^2}y = x^2 + x \]

, Exact solution is: \( y(x) = \frac{1}{18}x^3(-2x - 9 + 6(\ln x)x) + C_1x + C_2x^4 \)

Solution:

Cauchy-Euler equation, so characteristic equation is
\[ m^2 + (-4 - 1)m + 4 = 0 \Rightarrow m^2 - 5m + 4 = 0 \Rightarrow m = 1, 4. \]

\( y_h = C_1x + C_2x^4 \). Use Variation of Parameters to find \( y_p \).

\[ \Rightarrow y_p = v_1y_1' + v_2y_2' = v_1x + v_2x^4. \]

\( W[y_1,y_2] = y_1y_2' - y_2y_1' = x(4x^3) - x^4(1) = 3x^4. \)

3. (a) (10 pts) Find

\[ \mathcal{L}^{-1} \left( \frac{s + 4}{s^2 + 4s + 8} \right) \]

Solution:

\[ \mathcal{L}^{-1} \left( \frac{s + 4}{s^2 + 4s + 8} \right) = \mathcal{L}^{-1} \left( \frac{s + 4}{(s + 2)^2 + 4} \right) \]

\[ = \mathcal{L}^{-1} \left( \frac{s + 2}{(s + 2)^2 + 4} + \frac{2}{(s + 2)^2 + 4} \right) \]

\[ = e^{-2t} \cos 2t + e^{-2t} \sin 2t \]

(b) (15 pts) Use Laplace Transforms to solve:

\[ y'' - 3y' + 2y = e^{-x}, \quad y(0) = 1, \quad y'(0) = 1 \]

Solution:

\[ \mathcal{L} \left( y'' - 3y' + 2y \right) = \mathcal{L}(e^{-x}) = \frac{1}{s + 1} \]

Thus

\[ \left( s^2 - 3s + 2 \right) \mathcal{L}(y) - sy(0) - y'(0) + 3y(0) = \frac{1}{s + 1} \]

\[ \left( s^2 - 3s + 2 \right) \mathcal{L}(y) = \frac{1}{s + 1} + s - 2 \]
\[ \mathcal{L}(y) = \frac{1}{(s+1)(s-1)(s-2)} + \frac{s}{(s-1)(s-2)} - \frac{2}{(s-1)(s-2)} \]

\[ = \frac{-1 + s^2 - s}{(s+1)(s-1)(s-2)} \]

Thus

\[ y(x) = \frac{1}{2} e^x + \frac{1}{3} e^{2x} + \frac{1}{6} e^{-x} \]

4. (a) (12 pts) Find the eigenvalues and eigenfunctions for the problem

\[ y'' + \lambda y = 0; \quad y(0) = 0; \quad y'(\pi) = 0. \]

Be sure to consider the cases \( \lambda > 0, \lambda = 0, \) and \( \lambda < 0. \)

Solution:

Case I: \( \lambda < 0. \) Let \( \lambda = -k^2. \) Then \( r^2 - \lambda^2 = 0 \Rightarrow r = \pm k. \)

So \( y = C_1 e^{kx} + C_2 e^{-kx}. \quad y' = C_1 ke^{kx} - C_2 ke^{-kx} \)

\( y(0) = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2. \)

\( y'(\pi) = C_1 k e^{k\pi} + C_2 k e^{-k\pi} = 0 \Rightarrow C_1 = 0 \Rightarrow C_2 = 0 \Rightarrow y = 0. \)

Case II: \( \lambda = 0. \) \( y = C_1 x + C_2. \quad y' = C_1 \Rightarrow y'(\pi) = C_1 = 0. \)

\( y(0) = 0 \) implies that \( C_2 = 0, \) Thus \( y = 0, \) and 0 is not an eigenvalue.

Case III: \( \lambda = k^2 > 0. \) Then \( y = C_1 \sin kx + C_2 \cos kx. \)

\( y'(x) = C_1 k \cos kx - C_2 k \sin kx. \)

\( y(0) = C_2 k = 0 \Rightarrow C_2 = 0. \)

\( y'(\pi) = C_1 k \cos k\pi = 0 \Rightarrow C_1 \neq 0, \quad k\pi = (2n + 1) \cdot \frac{\pi}{2} \)

\( \Rightarrow k = \frac{2n+1}{2}, \quad n = 0, 1, 2, \ldots. \)

Therefore \( y_n(x) = C_n \sin\left(\frac{2n+1}{2}x\right), \quad \lambda_n = \frac{(2n+1)^2}{4}, \quad n = 0, 1, 2, \ldots \)

4(b) (13 pts) Find the power series solution to

\[ y'' - xy' + y = 0 \]

near \( x = 0. \) Be sure to give the recurrence relation. Indicate the two linearly independent solutions and give the first four nonzero terms the solution.

Solution:

Let \( y = \sum_{n=0}^{\infty} a_n x^n. \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \)

Plugging in gives

\[ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0. \]

Let \( k = n - 2 \) \( n = k + 2 \)

\[ \Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0. \]
\( (2)(1)a_2 + a_0 + (3)(2)a_3 x + \sum_{k=2}^{\infty}[(k+2)(k+1)a_{k+2} - (k-1)a_k]x^k = 0. \)

\[ a_2 = -\frac{a_0}{2}, \quad a_3 = 0 \]
and the Recurrence relation: 
\[ a_{k+2} = \frac{a_k(k-1)}{(k+1)(k+2)}, \quad k = 2, 3, \ldots \]

\[ k = 2 : a_4 = \frac{a_2}{3(4)} = -\frac{a_0}{24}, \quad k = 3 : a_5 = 0. \]

\[ y = a_0 \left(1 - \frac{x^2}{2} - \frac{x^4}{24} + \cdots\right) + a_1 x. \]

5. (a) (15 pts) Solve 
\[ y'' + y = \tan x \]

Solution: See Example 1 Page 215 of text.
The homogeneous solutions are \( \cos x \) and \( \sin x \). Thus
\[ y_p = v_1(x) \cos x + v_2(x) \sin x \]
The two equations for \( v_1' \) and \( v_2' \) are
\[ (\cos x)v_1' + (\sin x)v_2' = 0 \]
\[ (-\sin x)v_1' + (\cos x)v_2' = \tan x \]
Solving yields
\[ v_1' = -\tan x \sin x \]
\[ v_2' = \tan x \cos x = \sin x \]
Thus
\[ v_1 = -\int \tan x \sin x dx = \sin x - \ln|\sec x + \tan x| \]
\[ v_2 = \int \sin x dx = -\cos x \]
Thus
\[ y_p = -(\cos x) \ln|\sec x + \tan x| \]
Finally
\[ y = c_1 \cos x + c_2 \sin x - (\cos x) \ln|\sec x + \tan x| \]

(b) (10 pts) Compute the Wronskian for the solutions to the differential equation
\[ y'' - 2y' + y = 0 \]

Solution:
\( y'' - 2y' + y = 0 \), Exact solution is: \( y(x) = C_1 e^x + C_2 e^{2x} x \). Let \( y_1 = e^x \) and \( y_2 = xe^x \)
\[ W[y_1, y_2] = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \]

6. (a) (10 pts) Use separation of variables, \( u(x,t) = X(x)T(t) \), to find two ordinary differential
equations which $X(x)$ and $T(t)$ must satisfy to be a solution of

$$- 3x^2 t^4 \frac{\partial^2 u}{\partial x^2} + (x - 2)^4 (t + 6)^3 \frac{\partial^2 u}{\partial t^2} = 0.$$
X'(1) = bC_1 e^b + bC_1 e^{-b} = 0 = bC_1 (e^b - e^{-b}) \Rightarrow C_1 = 0 = C_2, i.e., a trivial case again.

Case 3. $K < 0, K = -\beta^2$.

$$r_{1,2} = \pm \beta i$$

$X(x) = C_1 \cos \beta x + C_2 \sin \beta x$

$X(0) = 0 = C_1 \cos 0; \ C_1 = 0 \Rightarrow X(x) = C_2 \sin \beta x.$

$X'(1) = C_2 \beta \cos \beta = 0, \ \beta = (2n + 1) \frac{\pi}{2} \Rightarrow$

$$K = -\beta^2 = -(2n + 1)^2 \frac{\pi^2}{4}$$

$$X_n(x) = a_n \sin \left[ (2n + 1) \frac{\pi}{2} x \right]$$

$n = 0, 1, 2, \ldots$

Now $T' - KT = 0$ $\iff$ $T' + (2n + 1)^2 \frac{\pi^2}{4} T = 0$ $\Rightarrow$ after solving the 1st order DE

$$T_n(t) = C_n e^{-\frac{\pi^2}{4} t(2n+1)^2}$$

$$u(x,t) = X(x)T(t)$$

$$u_n(x,t) = a_n \sin \left[ (2n + 1) \frac{\pi}{2} x \right] \left[ C_n e^{-\frac{\pi^2}{4} t(2n+1)^2} \right]$$

We include $C_n$ into $a_n$, obtaining $b_n$.

$$u_n(x,t) = b_n e^{-\frac{\pi^2}{4} t(2n+1)^2} \sin \left[ (2n + 1) \frac{\pi}{2} x \right]$$

Thus

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} b_n e^{-\frac{\pi^2}{4} t(2n+1)^2} \sin \left[ (2n + 1) \frac{\pi}{2} x \right]$$

Now, applying the IC,

$$u(x,0) = \sum_{n=0}^{\infty} b_n \sin \left[ (2n + 1) \frac{\pi}{2} x \right].$$

>From the IC $\Rightarrow u(x,0) = b_1 \sin \frac{\pi}{2} x + b_4 \sin \frac{9\pi}{2} x$, i.e., $b_0 = -3$ and $b_4 = 12$. All other $b_i$’s are equal to zero.

- Finally,
\[ u(x, t) = -3 \sin\left(\frac{\pi}{2} x\right)e^{-\frac{\pi^2 t}{4}} + 12 \sin\left(9 \frac{\pi}{2} x\right)e^{-\frac{81\pi^2 t}{4}} \]

**Part II: Choose any two (2) questions.**

7. (a) (13 pts) Consider the function

\[
f(x) = \begin{cases} 
  x & 0 \leq x \leq \frac{\pi}{2} \\
  \pi - x & \frac{\pi}{2} \leq x \leq \pi 
\end{cases}
\]

Write down the expression for the Fourier cosine series for this function. Give a formula for the coefficients in this expression with appropriate limits. **DO NOT EVALUATE THE COEFFICIENTS.**

**Solution:** \( L = \pi \)

\[
f(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos nx
\]

\[
b_0 = \frac{1}{\pi} \int_{0}^{\pi} f(x) dx
\]

\[
= \frac{1}{\pi} \left[ \int_{0}^{\frac{\pi}{2}} x dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) dx \right]
\]

\[
b_n = \frac{2}{\pi} \left[ \int_{0}^{\frac{\pi}{2}} x \cos nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \cos nx dx \right]
\]

(b) (12 pts) Sketch the graph of the function represented by the Fourier cosine series in 7 (a) on \(-\pi \leq x \leq 3\pi\).

\[
\frac{\pi}{2} = 1.5708, \pi = 3.1416, \frac{3\pi}{2} = 4.7124, 2\pi = 6.2832, \frac{5\pi}{2} = 7.8540, 3\pi = 9.4248
\]

We first graph the function on \([0, \pi]\). The graph is shown below.

\[
\begin{array}{c}
\text{y} \ 1.5 \\
1.0 \\
0.5 \\
0.0 \\
\text{0.0} \ 0.5 \ 1.0 \ 1.5 \ 2.0 \ 2.5 \ 3.0 \\
\text{x}
\end{array}
\]

Since this is a Fourier cosine expansion, the function is extended to \([-\pi, 0]\) by simply reflecting it...
across the $y$-axis. Thus the graph on $[-\pi, \pi]$ is $-x$

Since we now have the graph on $[-\pi, \pi]$ and the graph has period $2\pi$, we simply "pick up" the graph and move it to the right to get what looks like on $[\pi, 3\pi]$. Thus

8. (25 pts) Find the solution to the problem

$$u_{tt} = 100u_{xx}$$
$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = x(\pi - x)$$
$$u_t(x,0) = 0$$

Your answer must display all coefficients.
Note:  
\[ \int u^2 \sin au \, du = \frac{1}{a^2} \left(-a^2 u^2 \cos au + 2 \cos au + 2au \sin au\right) + C \]
\[ \int u \sin au \, du = \frac{1}{a^2} (\sin au - au \cos au) + C \]

Solution:
Assume 
\[ u(x, t) = X(x)T(t) \]

The the PDE implies 
\[ \frac{X''}{X} = \frac{T''}{100T} = k \]

Since the left hand side is a function of \(x\) only, and the right hand side is a function of \(t\) only. Thus we have two ODEs 
\[ X'' - kX = 0 \]
\[ T'' - 100kT = 0 \]

The BCs imply 
\[ X(0) = X(\pi) = 0 \]

and 
\[ T'(0) = 0 \]

The solution to the BVP for \(X\) is 
\[ X_n(x) = c_n \sin nx \]

with
\[ k = -n^2 \quad n = 1, 2, \ldots \]

The solution to the equation for \(T\) that satisfies the initial condition \(T'(0) = 0\) is 
\[ T_n(t) = d_n \cos 10nt \]

Hence 
\[ u_n(x, t) = A_n \sin nx \cos 10nt \quad n = 1, 2, \ldots \]

Thus 
\[ u(x, t) = \sum_{n=1}^{\infty} A_n \sin nx \cos 10nt \]

The last condition to be satisfied is 
\[ u(x, 0) = x(\pi - x) = \sum_{n=1}^{\infty} A_n \sin nx \]

Thus the \(A_n\)'s are the Fourier sine coefficients of this expansion and
\[ A_n = \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin nx \, dx \]
\[ = -\left( \frac{2}{\pi} \right) \left( \frac{2 \cos \pi n - 2}{n^3} \right) \]
\[ = \begin{cases} 
0 & \text{for } n \text{ even} \\
\frac{8}{\pi n^3} & \text{for } n \text{ odd}
\end{cases} \]

where the given formulas have been used to evaluate the integrals. Thus
\[ u(x, t) = \sum_{n=1}^{\infty} A_n \cos 10nt(\sin nx) \]
where
\[ A_n = \begin{cases} 
0 & \text{for } n \text{ even} \\
\frac{8}{\pi n^3} & \text{for } n \text{ odd}
\end{cases} \]

9. (a) (12 pts) Find the Laplace transform of
\[ f(x) = x \cos \sqrt{7} x \]
Solution: From the table
\[ \mathcal{L}(xf(x)) = -\frac{d}{ds} \mathcal{L}(f(s)) \]
\[ \mathcal{L}(\cos(\sqrt{7} t)) = \frac{s}{s^2 + 7} \]
\[ \frac{d}{ds} \left( \frac{s}{s^2 + 7} \right) = -\frac{s^2 - 7}{(s^2 + 7)^2} \]
Thus
\[ \mathcal{L}(x \cos \sqrt{7} x) = \frac{s^2 - 7}{(s^2 + 7)^2} \]

(b) (12 pts) Show that the equation
\[ \left( \frac{y^2}{2} + 2ye^x \right) dx + (y + e^x) dy = 0 \]
is not exact and then find an integrating factor for this equation. Do not solve the equation.
Solution:
\[ \frac{y^2}{2} + 2ye^x + (y + e^x)y' = 0 \]
\[ \Rightarrow \left( \frac{y^2}{2} + 2ye^x \right) dx + (y + e^x) dy = 0 \]
\[ M_y = y + 2e^x \quad N_x = e^x \quad \text{therefore not exact.} \]
\[ u \left( \frac{y^2}{2} + 2ye^x \right) dx + u(y + e^x) dy = 0 \]
\begin{align*}
    u_y \left( \frac{y^2}{2} + 2ye^x \right) dx + u(y + e^x) dy &= u_x(y + e^x) + u(e^x) \\
    u_y &= 0 \quad \Rightarrow \quad u(y + e^x) = u_x(y + e^x) \\
    u &= u_x \quad \Rightarrow \quad u = e^x \\
    \left( e^x \frac{y^2}{2} + 2ye^{2x} \right) dx + (ye^x + e^{2x}) dy &= 0 \\
    M_y &= ye^x + 2e^{2x} \\
    N_x &= ye^x + 2e^{2x}
\end{align*}
Table of Laplace Transforms

\[
\begin{align*}
  f(t) & \quad \hat{f}(s) \\
  t^{n-1} \frac{1}{(n-1)!} & \quad \frac{1}{s^n} \quad n \geq 1 \quad s > 0 \\
  \sin(ax) & \quad \frac{a}{s^2 + a^2} \quad s > a \\
  \cos(ax) & \quad \frac{s}{s^2 + a^2} \quad s > a \\
  e^{-bt}f(t) & \quad \hat{f}(s + b) \\
  t^n f(t) & \quad (-1)^n \frac{d^n}{ds^n} \hat{f}(s)
\end{align*}
\]