Ma 221

Review of Power Series

Please note that there is material on power series at Visual Calculus. Some of this material was used as part of the presentation of the topics that follow.

The Ratio Test

Recall

Definition: A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Ratio Test: Given a series $\sum a_n$, let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

(a) If $L < 1$, then the series converges absolutely.
(b) If $L > 1$, then the series is divergent.
(c) If $L = 1$, then the test fails (i.e., is inconclusive.)

Remark For an example of (c) consider $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. Both have a value of 1 for L. Yet, the former diverges, whereas the latter converges.

Example Determine whether

$$\sum_{n=1}^{\infty} \frac{(n+1)5^n}{n3^{2n}}$$

is convergent.
Solution: We apply the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+2)5^{n+1}}{(n+1)3^{2n+2}}}{\frac{(n+1)5^n}{n3^{2n}}} \right| = \lim_{n\to\infty} \left| \frac{(n+2)5^{n+1}}{(n+1)3^{2n+2}} \cdot \frac{n3^{2n}}{(n+1)5^n} \right| = \lim_{n\to\infty} \frac{5}{3^2} \left(\frac{n^2+2n}{n^2+2n+1} \right) = \frac{5}{9} < 1.$$

Therefore, the given series converges by the Ratio Test.
Example Test the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{n!}$$

for convergence or divergence.
Solution: Using the Ratio Test we have

$$L = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}4^{n+1}}{(n+1)!}}{\dfrac{(-1)^n 4^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}}{(n+1)!}\cdot\frac{n!}{4^n}\right| = \lim_{n\to\infty}\frac{4}{n+1} = 0$$

Thus this series converges absolutely, since $L < 1$.

What is a Power Series?

Recall that the geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\dfrac{a}{1-r}$ provided $|r| < 1$. In this section we will take the ratio r to be a variable x. In particular, the geometric series

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \cdots$$

is an example of a power series. It converges on the interval $-1 < x < 1$, which we say is centered at 0 and has radius 1. All this will be generalized in this section.

Definition A power series is a series of the form

$$S(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where $a, c_0, c_1, c_2, \ldots$ are constants. The numbers c_n are called the coefficients of the series and the number a is called the center of the series. We also say that the series is expanded about $x = a$ and that the n^{th}-order term is $c_n(x-a)^n$.

Remark In the 0^{th}-order term $c_0(x-a)^0 = c_0$ we follow the convention that $(x-a)^0 = 1$, even when $x = a$, in spite of the fact that 0^0 is normally undefined. All the remaining terms go to 0 when $x = a$, since $(a-a)^n = 0^n = 0$ for $n \neq 0$. Therefore $S(a) = c_0$.

Remark Very often we consider power series centered at $x = 0$ which have the simpler form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Example For the series

$$\sum_{n=0}^{\infty} \frac{3(n+1)^2}{2^n}(x-5)^n = 3 + 6(x-5) + \frac{27}{4}(x-5)^2 + 6(x-5)^3 + \cdots$$

identify the center and the coefficient of the 4^{th}-order term.

Solution: The center is at $x = 5$. The coefficient of the n^{th}-order term is $c_n = \dfrac{3(n+1)^2}{2^n}$. So the coefficient of 4^{th}-order term is

$$c_4 = \frac{3(4+1)^2}{2^4} = \frac{75}{16}$$

Exercise For the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{2}{n+2}(x+4)^n$$

1. Identify the center.

-2 2 -4 4

2. Identify the coefficient of the 3rd-order term.

$\frac{2}{5}$ $-\frac{2}{5}$ $\frac{1}{2}$ $-\frac{1}{2}$

3. Write out the terms of the series up to 3rd-order....................................

Radius of Convergence

Recall that the geometric series

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \cdots$$

converges on the interval $-1 < x < 1$, which we say is centered at 0 and has radius 1. For a general power series, we have:

Theorem (**Power Series Convergence Theorem**) The convergence of a power series

$$S(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is characterized by one of the following three cases:

1. There is a positive number R, called the radius of convergence, such that the series $S(x)$ converges absolutely on the interval $(a-R, a+R)$. The series may also converge at one, both or neither of the endpoints $a-R$ and $a+R$.

2. The series $S(x)$ converges only for $x = a$. In this case, we say that the radius of convergence is $R = 0$.

3. The series $S(x)$ converges for all real numbers x. In this case, we say that the radius of convergence is $R = \infty$.

Note that a Power Series always converges for $x = a$, since at $x = a$ $\sum_{n=0}^{\infty} c_n x^n = c_0$. It may or may not converge for other values of x.

Remark Notice that the center of the interval $(a-R, a+R)$ is at a, which is the center of the series. Further, the interval $(a-R, a+R)$ can also be specified by either of the triple inequalities

$$a - R < x < a + R \qquad \text{or} \qquad -R < x - a < R$$

or by the absolute value inequality

$$|x-a| < R.$$

Remark **The radius of convergence is usually found by applying the ratio test to the series.**

Example Find the center and radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)^2}{2^n}(x-5)^n$.

Solution: The center is $a = 5$. To find the radius we apply the ratio test

$$a_n = \frac{(n+1)^2}{2^n}(x-5)^n \quad \text{and} \quad a_{n+1} = \frac{(n+2)^2}{2^{n+1}}(x-5)^{n+1}$$

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(n+2)^2}{2^{n+1}}(x-5)^{n+1} \frac{2^n}{(n+1)^2} \frac{1}{(x-5)^n} \right| \\ &= \lim_{n\to\infty} \left(\frac{n+2}{n+1} \right)^2 \frac{|x-5|}{2} = \frac{|x-5|}{2} \left(\lim_{n\to\infty} \frac{n+2}{n+1} \right)^2 \\ &= \frac{|x-5|}{2} \end{aligned}$$

The ratio test says the series converges absolutely if $L < 1$. In other words, $\frac{|x-5|}{2} < 1$ or $|x-5| < 2$, from which we identify the radius of convergence as $R = 2$. Thus, the series converges absolutely on the interval $(3, 7)$.

Now you do it:

Exercise Find the center and radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} (-3)^n (x-2)^n$. ...

The example and exercise above illustrate the first case in the Power Series Convergence Theorem. Here are some examples of the other two cases and of the use of the ratio test:

Example Find the center and radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}(x+1)^n$.

Solution: The center is $a = -1$. To find the radius we apply the ratio test:

$$|a_n| = \frac{2^n}{n!}|x+1|^n \quad \text{and} \quad |a_{n+1}| = \frac{2^{n+1}}{(n+1)!}|x+1|^{n+1}$$

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}|x+1|^{n+1} \frac{n!}{2^n} \frac{1}{|x+1|^n} \\ &= \lim_{n\to\infty} \frac{n!}{(n+1)!} 2|x+1| = \lim_{n\to\infty} \frac{1}{(n+1)} 2|x+1| \\ &= 0 \quad \text{for all } x. \end{aligned}$$

The ratio test says the series converges absolutely if $L < 1$. Since $L = 0$, the series converges for all x and the radius of convergence is $R = \infty$.

Example Find the center and radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n}(x+1)^n$.

Solution: The center is again $a = -1$. To find the radius we apply the ratio test:

$$|a_n| = \frac{n!}{2^n}|x+1|^n \quad \text{and} \quad |a_{n+1}| = \frac{(n+1)!}{2^{n+1}}|x+1|^{n+1}$$

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}}|x+1|^{n+1} \frac{2^n}{n!} \frac{1}{|x+1|^n} \\ &= \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{1}{2}|x+1| = \lim_{n\to\infty} \frac{(n+1)}{2}|x+1| \\ &= \begin{cases} \infty & \text{for all } x \text{ except } x = -1 \\ 0 & \text{for } x = -1. \end{cases} \end{aligned}$$

The ratio test says the series diverges if $L > 1$ and converges if $L < 1$. So, the series diverges for all x except $x = -1$ and converges for $x = -1$. Thus, the radius of convergence is $R = 0$.

And some more exercises:

Exercise Find the center and radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(x-3)^n$................

...

Exercise Find the center and radius of convergence of the series $\sum_{n=0}^{\infty} \frac{n!}{2^n}(x-1)^n$.. ...

Interval of Convergence

Definition The Power Series Convergence Theorem implies that a power series

$$S(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

converges on an interval called its interval of convergence. This interval may consist of a single point $[a]$, the set of all real numbers $(-\infty, \infty)$, or a finite interval which may be open: $(a-R, a+R)$, closed: $[a-R, a+R]$, or half open: $[a-R, a+R)$ or $(a-R, a+R]$.

We have seen that the center a may be read off the series, and the radius R may be determined using the ratio test (or the root test). It remains to determine the convergence at the endpoints of the interval of convergence.

Remark It is much more important to be able to determine the radius of convergence than it is to be able to determine whether the series converges at the endpoints of the interval of convergence.

Remark You **cannot use the ratio test or the root test** to determine the convergence at the endpoints, because these tests fail when $L = 1$ which is precisely at the endpoints of the interval of convergence. You **must use some other test**.

The following three examples illustrate the three cases of an open, closed or half open interval of convergence.

Example Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{2}{3^n}(x-1)^n$........................... ...

Example Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{2}{3^n n^2}(x-1)^n$....................... ...

You try this one first:

Example Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{2}{3^n \sqrt{n}}(x-1)^n$..................... ...

Example For what values of x is the series

$$\sum_{n=1}^{\infty} \frac{n(2x-1)^n}{4^n} = \frac{(2x-1)}{4} + \frac{2(2x-1)^2}{16} + \frac{3(2x-1)^3}{64} + \cdots$$

convergent?

We shall use the Ratio Test again.

$$L = \lim_{n\to\infty}\left|\frac{(n+1)(2x-1)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n(2x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{2x-1}{4}\right|$$

.

For the series to converge we must have $L < 1$, that is $\left|\frac{2x-1}{4}\right| < 1$ or equivalently $\left|\frac{1}{2}x - \frac{1}{4}\right| < 1$. Thus

$$-1 < \frac{x}{2} - \frac{1}{4} < 1$$

or

$$-\frac{3}{4} < \frac{x}{2} < \frac{5}{4}$$

or finally

$$-\frac{3}{2} < x < \frac{5}{2}$$

Hence, the series converges if $-\frac{3}{2} < x < \frac{5}{2}$ and diverges for $x > \frac{5}{2}$ and $x < -\frac{3}{2}$. The cases when $x = -\frac{3}{2}$ and $x = \frac{5}{2}$ must be tested separately.

When $x = -\frac{3}{2}$, then $\sum_{n=1}^{\infty} \frac{n(2x-1)^n}{4^n} = \sum_{n=1}^{\infty} \frac{n(-4)^n}{4^n} = \sum_{n=1}^{\infty} n(-1)^n$, which diverges since the n th term of this series does not go to zero as $n \to \infty$.

When $x = \frac{5}{2}$, then $\sum_{n=1}^{\infty} \frac{n(2x-1)^n}{4^n} = \sum_{n=1}^{\infty} \frac{n(4)^n}{4^n} = \sum_{n=1}^{\infty} n$, which again diverges. Why?

Thus this series converges in $-\frac{3}{2} < x < \frac{5}{2}$ and this is the Interval of Convergence. For this example

$R = 2.$

Exercise Find the interval of convergence and the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{3^n}{n} (x-1)^n$$