Ma 227 Final Exam 10 May 1995

Directions: This examination is in two parts. In Part I you must answer all six (6) problems. In Part II choose any two questions. Indicate on your blue book(s) which questions you have chosen.

Problem 1

a) (8 points)

Find the first four nonzero terms of the Fourier *sine* series of

$$f(x) = \begin{cases} 0 & 0 < x < \pi \\ -2 & \pi < x < 2\pi \end{cases}$$

Solution:

If $f(x)$ is a function defined on $[0, L]$, then its Fourier sine expansion is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \text{ where } a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Here $L = 2\pi$ so that $f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{nx}{2}\right)$ and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin\frac{nx}{2}\,dx$$

Hence $a_n = \frac{1}{\pi}\left[\int_0^{\pi} 0 \sin\frac{nx}{2}\,dx + \int_{\pi}^{2\pi}(-2)\sin\frac{nx}{2}\,dx\right] =$

$$\frac{1}{\pi}(4)\left[\frac{\cos\pi n - \cos(\frac{1}{2}\pi n)}{n}\right].$$

Evaluating this last expression for $n = 1, 2, 3, 4, 5$ we get

$n = 1 \quad a_1 = \frac{4}{\pi}[-1]$

$n = 2 \quad a_2 = \frac{8}{\pi}[1]$

$n = 3 \quad a_3 = \frac{12}{\pi}\left[-\frac{1}{3}\right] = -\frac{4}{\pi}$

$n = 4 \quad a_4 = \frac{16}{\pi}0 = 0$

$n = 5 \quad a_5 = \frac{20}{\pi}[-\frac{1}{5}] = -\frac{4}{\pi}$

Thus $f(x) = -\frac{4}{\pi}\sin\frac{x}{2} + \frac{8}{\pi}\sin x - \frac{4}{\pi}\sin\frac{3x}{2} + 0\sin 2x - \frac{4}{\pi}\sin\frac{5x}{2} + \cdots$

b) (8 points)

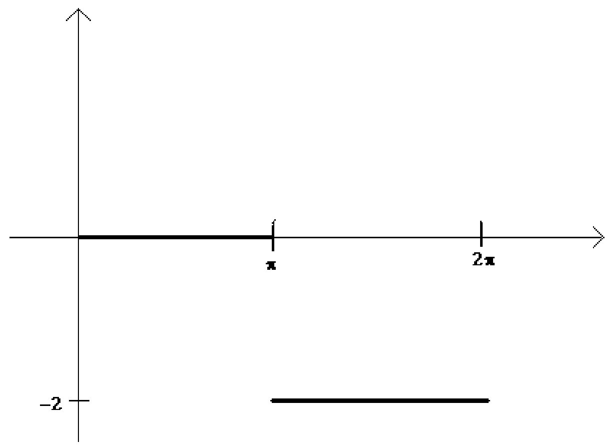
Sketch the graph of the function to which the Fourier sine series of the function

$$f(x) = \begin{cases} 0 & 0 < x < \pi \\ -2 & \pi < x < 2\pi \end{cases}$$

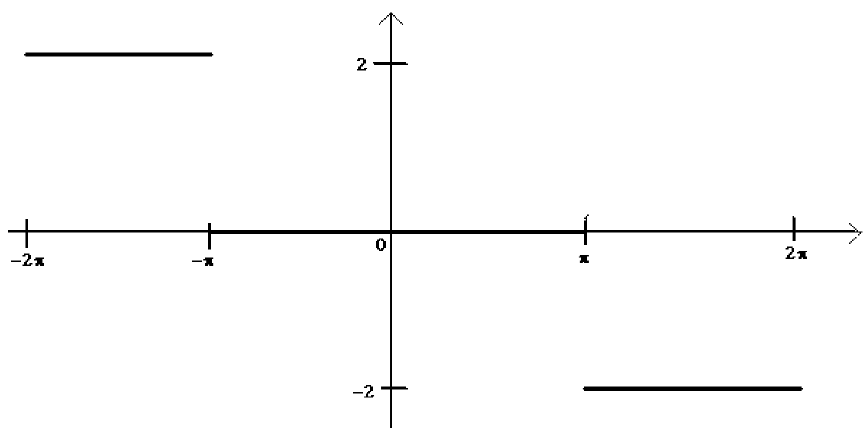
converges on $-2\pi < x < 4\pi$.

Solution

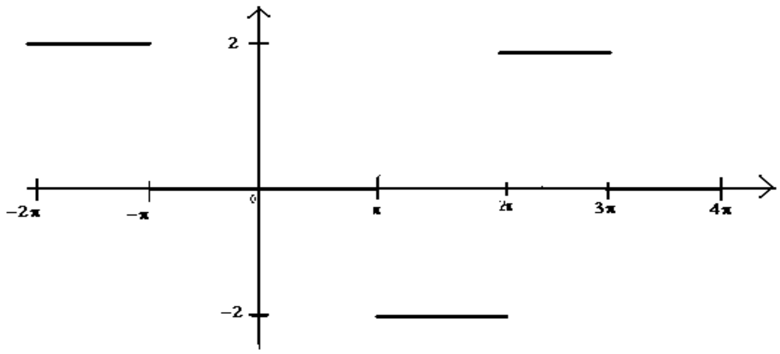
The graph of the given function is below.



Since we were asked to find the Fourier sine expansion of $f(x)$, this means that we are seeking an odd expansion of f. Hence the graph above is reflected first across the $y-axis$, and then across the $x-axis$ to get an odd function. The result is given below.



The Fourier sine series generates an odd function with period $2L$. Here $L = 2\pi$, so the function generated by the Fourier series has period $2(2\pi) = 4\pi$. Since the last graph above given the function on the interval $[-2\pi, 2\pi]$, i.e., on an interval of length 4π, we may move this graph either to the left or the right to get the function anywhere. Thus we have



c) **(9 points)**

Find the eigenvalues and eigenfunctions for the problem

$$y'' + \lambda y = 0, \quad y'(0) = y'(1) = 0$$

Be sure to check the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

Solution

I. Consider the case $\lambda < 0$ first. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. The DE becomes

$$y'' - \alpha^2 y = 0.$$

The general solution of this equation is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. Thus

$$y'(x) = c_1 \alpha e^{\alpha x} - c_2 \alpha e^{-\alpha x}$$

$$y'(0) = c_1 \alpha - c_2 \alpha = 0 \text{ and } y'(1) = c_1 \alpha e^{\alpha} - c_2 \alpha e^{-\alpha} = 0.$$

The first equation implies that $c_1 = c_2$. Thus the second equation becomes $c_1(e^{\alpha} - e^{-\alpha}) = 0$. Thus $c_1 = 0$, this tells us that $c_2 = 0$ also. Therefore $y = 0$ is the only solution if $\lambda < 0$.

II. Suppose $\lambda = 0$. The DE becomes $y'' = 0$ which has the solution $y = c_1 x + c_2$. The boundary conditions imply $c_1 = 0$, so that $y = c_2$. Thus $y = c_2$ where $c_2 \neq 0$ is an eigenfunction corresponding to the eigenvalue $\lambda = 0$.

III. Suppose $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE becomes

$$y'' + \beta^2 y = 0.$$

The general solution of this equation is $y(x) = c_1 \sin \beta x + c_2 \cos \beta x$. Thus

$$y'(x) = c_1 \beta \cos \beta x - c_2 \beta \sin \beta x$$

Now $y'(0) = c_1 \beta = 0$ Since $\beta \neq 0$, we must have $c_1 = 0$. Thus $y(x) = c_2 \cos \beta x$. Now $y'(x) = -c_2 \beta \sin \beta x$ and $y'(1) = -c_2 \beta \sin \beta = 0$. For a nontrivial solution we must have $c_2 \neq 0$. This means that $\sin \beta = 0$ or $\beta = n\pi$, $n = 1, 2, 3, \ldots$ The eigenvalues are therefore $\lambda = \beta^2 = n^2 \pi^2$ and the corresponding eigenfunctions are $y_n = a_n \cos n\pi x$, $n = 1, 2, 3, \ldots$

We may also include the eigenfunction found in II above by allowing n to equal 0. Hence all of the eigenfunctions are given by $y_n = a_n \cos n\pi x,\ n = 0, 1, 2, 3, \ldots$ with corresponding eigenvalues $\lambda = n^2\pi^2,\ n = 0, 1, 2, 3, \ldots$

Problem 2

a) (10 points)

Use separation of variables, $u(x,t) = X(x)T(t)$, to find ordinary differential equations which $X(x)$ and $T(t)$ must satisfy if $u(x,t)$ is to be a solution of

$$11t^2x^9u_{xx} - (t-3)(x+2)u_{ttt} = 0$$

Solution

$$u_x = X'T, \qquad u_{xx} = X''T, \qquad\qquad u_t = XT', \text{ etc.}$$

Thus the given equation becomes

$$11t^2x^9X''T - (t-3)(x+2)XT''' = 0$$

$$\Rightarrow \qquad 11x^9\frac{X''}{(x+2)X} = (t-3)\frac{T'''}{t^2T} = k, \qquad k \text{ a constant}$$

This yields the two ODEs

$$11x^9X'' - k(x+2)X = 0$$

$$(t-3)T''' - kt^2T = 0$$

b) (15 points)

Solve:

P.D.E.: $u_{xx} - 4u_{tt} = 0$

B.C.'s: $u_x(0,t) = 0 \qquad u_x(\pi,t) = 0$

I.C.'s: $u(x,0) = 0 \qquad u_t(x,0) = -8\cos(4x) + 17\cos(8x)$

Solution

Let $u(x,t) = X(x)T(t)$. Then differentiating and substituting in the PDE yields

$$X''T = 4XT''$$
$$\Rightarrow \qquad \frac{X''}{X} = 4\frac{T''}{T}$$

Using the argument that the left hand side is purely a function of x and the right hand side is purely a function of t, and the only way that they can be equal is if they are equal to a constant, we get

$$\frac{X''}{X} = 4\frac{T''}{T} = k \qquad k \text{ a constant}$$

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T'' - \frac{1}{4}kT = 0$$

The boundary condition $u_x(0,t) = 0$ implies, since $u_x(x,t) = X'(x)T(t)$ that $X'(0)T(t) = 0.$ We cannot have $T(t) = 0,$ since this would imply that $u(x,t) = 0.$ Thus $X'(0) = 0.$ Similarly, the boundary condition $u_x(\pi,t) = 0$ leads to $X'(\pi) = 0.$

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X'(0) = X'(\pi) = 0$$

This boundary value problem is very similar to the one given in Problem 1(c) above. (Its solution was discussed in the slide show Eigenvalues and Eigenfunctions for Boundary Value Problems.) The solution is

$$k = -n^2 \qquad X_n(x) = a_n \cos nx \qquad n = 1,2,3,\ldots$$

Substituting the values of k into the equation for $T(t)$ leads to

$$T'' + \frac{n^2}{4}T = 0$$

which has the solution $T_n(t) = b_n \sin\frac{nt}{2} + c_n \cos\frac{nt}{2},\ n = 1,2,3,\ldots$

The initial condition $u(x,0) = 0$ implies $X(x)T(0) = 0$ so that $T(0) = 0.$ Thus $c_n = 0.$

We now have the solutions

$$u_n(x,t) = A_n \cos nx \sin\frac{nt}{2} \qquad n = 1,2,3,\ldots$$

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \cos nx \sin\frac{nt}{2}$$

satisfies the PDE, the boundary conditions, and the first initial condition. Since

$$u_t(x,t) = \sum_{n=1}^{\infty} A_n\left(\frac{n}{2}\right)\cos nx \cos\frac{nt}{2}$$

the last initial condition leads to

$$u_t(x,0) = -8\cos(4x) + 17\cos(8x) = \sum_{n=1}^{\infty} A_n\left(\frac{n}{2}\right)\cos nx.$$

Matching the cosine terms on both sides of this equation leads to

$A_4\left(\frac{4}{2}\right) = -8$ so that $A_4 = -4$ and $A_8\left(\frac{8}{2}\right) = 17$ so that $A_8 = \frac{17}{4}.$ All of the other constants must be zero, since there are no cosine terms on the left to match with. Thus

$$u(x,t) = -4\cos 4x \sin\frac{4t}{2} + \frac{17}{4}\cos 8x \sin\frac{8t}{2} = -4\cos 4x \sin 2t + \frac{17}{4}\cos 8x \sin 4t$$

Problem 6

a) (15 points)

Verify Stokes' Theorem for the vector $\vec{v} = y\vec{i} - x\vec{j}$, where S is the hemisphere $x^2 + y^2 + z^2 = 9, z \geq 0$.

SOLUTION

We shall use the outward normal $\vec{N}$. We calculate $\oint_{\partial S} \vec{F} \cdot d\vec{r}$ first. Now ∂S is the circle $x^2 + y^2 = 9,\ z = 0$. We parametrize this as

$$x = 3\cos t, \qquad y = 3\sin t, \qquad z = 0 \qquad 0 \leq t \leq 2\pi$$

$$\vec{F} = 3\sin t\,\vec{i} - 3\cos t\,\vec{j}$$
$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k} = 3\cos t\,\vec{i} + 3\sin t\,\vec{j} + 0\vec{k} \quad \Rightarrow \quad \vec{r}\,'(t) = -3\sin t\,\vec{i} + 3\cos t\,\vec{j}$$

Thus, $\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-9\sin^2 t - 9\cos^2 t)dt = \int_0^{2\pi} (-9)dt = -18\pi$

Now consider $\iint_S curl\,\vec{F} \cdot \vec{N}\,ds$.

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\vec{k}$$

S is the surface $x^2 + y^2 + z^2 = 9 \qquad z \geq 0$. In spherical coordinates, $\rho = 3 \Rightarrow$

$x = 3\sin\phi\cos\theta, \qquad y = 3\sin\phi\sin\theta, \qquad z = 3\cos\phi \qquad$ Let $u = \phi \qquad v = \theta$

and therefore $\vec{r}(u,v) = 3\sin u\cos v\,\vec{i} + 3\sin u\sin v\,\vec{j} + 3\cos u\,\vec{k}$

$$\vec{r_u} \times \vec{r_v} = 9\sin^2 u\cos v\,\vec{i} + 9\sin^2 u\sin v\,\vec{j} + 9\sin u\cos u\,\vec{k}$$

At $\phi = \frac{\pi}{2}, \qquad \theta = 0,$ i.e., $u = \frac{\pi}{2} \qquad v = 0 \Rightarrow$

$\vec{r_u} \times \vec{r_v} = 9\vec{i}$, which is outward. Hence $\vec{N} = \vec{r_u} \times \vec{r_v}$ is outward.

Now $curl\,\vec{F} \cdot \vec{N} = -9\sin u\cos u$

$\iint_S curl\,\vec{F} \cdot \vec{N}ds = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (-18\sin u\cos u)dudv = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (-9\sin 2u)dudv = -18\pi$, as before. QED.

b) (10 points)

Find the value of the line integral $\oint_C xdx + (x+y)dy + (x+y+z)dz$, where C is the line segment from $(1,0,-1)$ to $(2,3,4)$.

SOLUTION

The line segment given is in 3D. We have to parametrize it first.

$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0} = t$ (these equalities mean simply that the slope is the same for x, y, and z).

That is, $\frac{x-1}{1} = \frac{y-0}{3} = \frac{z+1}{5} = t$

Consequently, $x = t+1,\ \ y = 3t,\ \ z = 5t-1,$ which is our parametrization.

Now, $dx = dt,\ \ dy = 3dt,\ \ dz = 5dt.$ Using, for example, that x goes from 1 to 2, we determine that

the parameter t goes from 0 to 1.

The integral becomes:

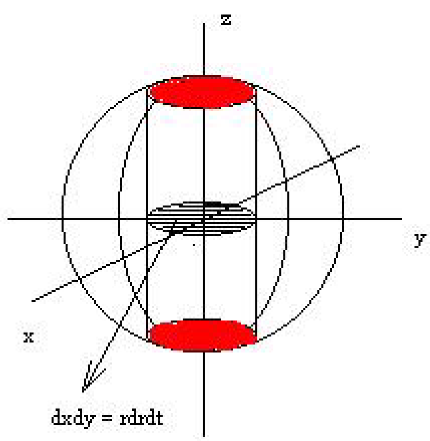
$$\oint_C xdx + (x+y)dy + (x+y+z)dz = \int_0^1 (t+1)dt + 3(t+1+3t)dt + 5(t+1+3t+5t-1)dt =$$
$$= \int_0^1 (t+1+3t+3+9t+5t+5+15t+25t-5)dt = \int_0^1 (58t+4)dt = 33$$

Problem 7

a) (10 points)

Find the surface area of the caps cut from the sphere $x^2+y^2+z^2 = 4$ by the cylinder $x^2+y^2 = 1$. Sketch the surface.

SOLUTION



$$A = \iint_{A(x,y)} \sqrt{1+f_x^2+f_y^2}\, dxdy$$

$$f(x,y) = \sqrt{4-x^2-y^2}$$

$$f_x = \frac{\partial\sqrt{4-x^2-y^2}}{\partial x} = -\frac{x}{\sqrt{(4-x^2-y^2)}} \quad \therefore\ f_x^2 = \frac{x^2}{4-x^2-y^2}$$

$$f_y = \frac{\partial\sqrt{4-x^2-y^2}}{\partial y} = -\frac{y}{\sqrt{(4-x^2-y^2)}} \quad \therefore\ f_y^2 = \frac{y^2}{4-x^2-y^2}$$

The integrand becomes: $\sqrt{1+f_x^2+f_y^2} = \sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}} = \sqrt{\frac{4}{4-x^2-y^2}}$

Now it is useful to switch to polar coordinates to make life easier: $x = r\cos\theta,\ \ y = r\sin\theta$. $dxdy = rdrd\theta$.

The integrand is, then, $\sqrt{\frac{4}{4-x^2-y^2}} = \sqrt{\frac{4}{4-r^2}}$. The limits of integration are: r from 0 to 1 (limits set by the cylinder!) and θ from 0 to 2π.

And the integral becomes

$$A = \int_0^{2\pi}\int_0^1 \sqrt{\frac{4}{4-r^2}}\, rdrd\theta = -4\pi\sqrt{3} + 8\pi.$$

We should not forget, though, that the above area is of one cap only, and has to be taken *twice*, so the answer is

$$A_{total} = 2A = 2(-4\pi\sqrt{3} + 8\pi) = 16\pi - 8\pi\sqrt{3}.$$

b. (15 points)

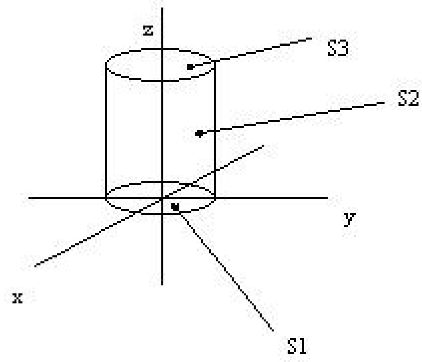
Let S be the surface of the solid cylinder T bounded by $z = 0$ and

$z = 3$ and $x^2 + y^2 = 4$. Evaluate $\int\int_S \vec{F} \bullet \vec{n} dS$, where

$\vec{F} = (x^2 + y^2 + z^2)(x\vec{i} + y\vec{j} + z\vec{k})$ and $\vec{n}$ is the outward unit normal.

Sketch the surface.

SOLUTION



S is composed of $S_1, S_2,$ and S_3.

On S_1 $\vec{n} = -\vec{k} \Rightarrow \vec{F} \bullet \vec{n} = -z(x^2 + y^2 + z^2)$.
But $z = 0$ on $S_1 \Rightarrow \vec{F} \bullet \vec{n} = 0 \Rightarrow \int\int_{S_1} \vec{F} \bullet \vec{n} ds = 0$.

On S_3 $z = 3$, $\vec{n} = +\vec{k} \Rightarrow \vec{F} \bullet \vec{n} = +z(x^2 + y^2 + z^2) = 3(x^2 + y^2 + 9) = 3x^2 + 3y^2 + 27$.
$ds = dxdy \Rightarrow$ introduce polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $ds = dxdy = rdrd\theta$.
$\int\int_{S_3} \vec{F} \bullet \vec{n} ds = \int\int_{S_3} (3x^2 + 3y^2 + 27) dxdy = \int_0^{2\pi}\int_0^2 (3r^2\sin^2\theta + 3r^2\cos^2\theta + 27) rdrd\theta =$
132π

On S_2 we shall use cylindrical coordinates $x = r\cos\theta \quad y = r\sin\theta \quad z = z$
Since our cylinder is $x^2 + y^2 = 4 \Rightarrow r = 2 \Rightarrow$

$\vec{r} = 2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k}$ where $0 \le z \le 3$.
Taking $u = \theta \quad v = z$ here, we have

$$\vec{r_\theta} = -2\sin\theta\vec{i} + 2\cos\theta\vec{j} \qquad \vec{r_z} = \vec{k}$$

$\Rightarrow \quad \vec{r_\theta} \times \vec{r_z} = 2\cos\theta\vec{i} + 2\sin\theta\vec{j} \Rightarrow |\vec{r_\theta} \times \vec{r_z}| = 2$
Thus $\vec{n} = \cos\theta\vec{i} + \sin\theta\vec{j}$. This is outward.

$\vec{F} \bullet \vec{n} = (4\cos^2\theta + 4\sin^2\theta + z^2)(2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k}) \bullet (\cos\theta\vec{i} + \sin\theta\vec{j}) =$
$= (4 + z^2)(2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k}) \bullet (\cos\theta\vec{i} + \sin\theta\vec{j}) =$
$= (4 + z^2)(2\cos^2\theta + 2\sin^2\theta) = 8 + 2z^2$.
Hence $\int\int_{S_2} \vec{F} \bullet \vec{n} ds = \int_0^{2\pi}\int_0^3 2(8 + 2z^2) dzd\theta = 168\pi$

Thus we have finally

$$\iint_S \vec{F} \cdot \vec{n} ds = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3}\right) \vec{F} \cdot \vec{n} ds = 0 + 132\pi + 168\pi = 300\pi.$$

Problem 8

a) **(13 points)**

Let S be the surface of the region V bounded by $z = 0$, $y = 0$, $y = 2$, and the parabolic cylinder $z = 1 - x^2$. Apply the divergence theorem to

compute $\iint_S \vec{F} \cdot \vec{n} dS$, where $\vec{n}$ is the outer unit normal to S and

$\vec{F} = (x + \cos y)\vec{i} + (y + \sin z)\vec{j} + (z + e^x)\vec{k}$.

SOLUTION

The Divergence Theorem states that $\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V div\vec{F} dV$.
$div\vec{F} = \nabla \cdot \vec{F} = 3$.

$$\therefore \iint_S \vec{F} \cdot \vec{n} dS = \int_{-1}^{1} \int_0^2 \int_0^{1-x^2} 3 dz dy dx = 8.$$

b) **(12 points)**

Find the volume of the region T that is bounded by the parabolic cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$.

SOLUTION

z goes from 0 to the plane $1 - x$
y goes from $-\sqrt{x}$ to $+\sqrt{x}$
x goes from 0 to 1.

$$\therefore V = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{1-x} dz dy dx = \frac{8}{15}.$$

Problem 9

a) **(15 points)**

Find the eigenvalues of the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Find the eigenvectors corresponding to the *largest* and *smallest* eigenvalues.

SOLUTION

$$\det\begin{bmatrix} 2-r & 0 & 0 \\ 1 & -r & 2 \\ 0 & 0 & 3-r \end{bmatrix} = 0, \text{ to find the eigenvalues } r.$$

When we expand the determinant and solve for r, we obtain the eigenvalues: $0, 2, 3$.

We have to find the eigenvectors corresponding to the first and the last of these.

Substitute 0 for r in the above matrix and multiply on the right by the desired eigenvector (unknown).

$$\begin{bmatrix} 2-r & 0 & 0 \\ 1 & -r & 2 \\ 0 & 0 & 3-r \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and we find the eigenvector } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Do the same for the eigenvalue 3.

$$\begin{bmatrix} 2-r & 0 & 0 \\ 1 & -r & 2 \\ 0 & 0 & 3-r \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and we find the eigenvector } \begin{bmatrix} 0 \\ 1 \\ \frac{3}{2} \end{bmatrix}$$

b) (**10 points**)

Let $\vec{F}$ be such that $\oint_C \vec{F} \bullet \vec{dr} = 0$ for any closed path C. Prove that $\int \vec{F} \bullet \vec{dr}$ is path-independent.

PROOF

Take two arbitrary distinct points $A \neq B$ on the closed path C. Thus C is divided into two parts.

Then $\oint_C \vec{F} \bullet \vec{dr} = \int_A^B \vec{F} \bullet \vec{dr} + \int_B^A \vec{F} \bullet \vec{dr} = 0$

Both integrals go, say, counterclockwise, but the first integral goes from A to B along the first part of C,

while the second goes from B back to A along the second part of C.

$\therefore\ I_{AB}^{I} = \int_A^B \vec{F} \bullet \vec{dr} = -\int_B^A \vec{F} \bullet \vec{dr} = I_{BA}^{II}$ Now lets switch the limits of the right-hand side integral.

$\therefore\ I_{AB}^{I} = \int_A^B \vec{F} \bullet \vec{dr} = +\int_A^B \vec{F} \bullet \vec{dr} = -I_{BA}^{II} = I_{AB}^{II}$.

So we obtain that the integrals are the same along *two different paths*, which we chose to be arbitrary

by arbitrarily choosing the points A and B.

Consequently, $\int \vec{F} \bullet \vec{dr}$ is path-independent; it doesn't matter which way we go from A to B.