Solutions: 1996 Ma 227 Final Exam

1. a) Find the first four non-zero terms on the Fourier cosine series of

$$f(x) = \begin{cases} 3 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

Cosine Formula: $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

$$a_0 = 1\left[\int_0^1 3dx + \int_1^2 0dx \right] = 3$$

$$a_n = \int_0^1 3\cos \frac{n\pi x}{2} dx + \int_1^2 0\cos \frac{n\pi x}{2} dx = \frac{6}{n\pi} \sin \frac{n\pi x}{2} \Big|_0^1 = \frac{6}{n\pi} \sin \frac{n\pi}{2}$$

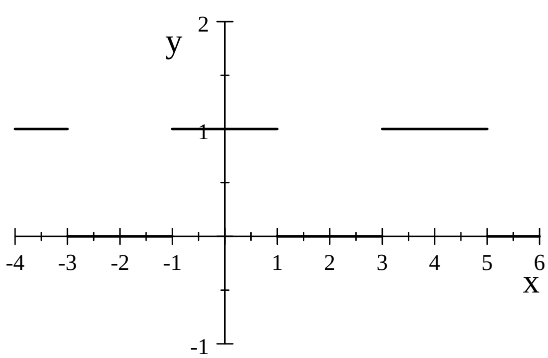
$$a_n = \begin{cases} \frac{6}{n\pi}(-1)^{\frac{n-1}{2}} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Thus $f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{2} = \sum_{n=1}^{9} \frac{6(-1)^{n+1}}{(2n-1)\pi} \cos \frac{(2n-1)\pi x}{2}$

computing the first few terms:

$f(x) = \frac{3}{2} + \frac{6}{\pi} \cos \frac{1}{2}\pi x - \frac{2}{\pi} \cos \frac{3}{2}\pi x + \frac{6}{5\pi} \cos \frac{5}{2}\pi x - \frac{6}{7\pi} \cos \frac{7}{2}\pi x + \frac{2}{3\pi} \cos \frac{9}{2}\pi x$

1. b) Sketch the graph of $f(x)$ on $-4 < x < 6$



1. c) Solve the boundary value problem:

$$y''(x) - y(x) = x; \qquad y(0) = 0; \qquad y'(1) = 1$$

homogeneous solution: $y''(x) - y(x) = 0$

characteristic equation: $r^2 - 1 = 0 \Rightarrow r = \pm 1$

$$y(x) = c_1 e^x + c_2 e^{-x}$$

particular solution: $\left.\begin{aligned} y(x) &= Ax + B \\ y'(x) &= A \\ y''(x) &= 0 \end{aligned}\right\} \Rightarrow A = -1 \Rightarrow y(x) = -x$

general solution: $y(x) = c_1 e^x + c_2 e^{-x} - x$ then $y'(x) = c_1 e^x - c_2 e^{-x} - 1$

B.C. $\Rightarrow$ $y(0) = c_1 e^0 + c_2 e^{-0} - 0 = 0 \Rightarrow c_1 = -c_2$

and $y'(1) = c_1 e^1 - c_2 e^{-1} - 1 = 1 \Rightarrow c_1 e - c_2 e^{-1} = 2$

$\begin{aligned} c_1 &= -c_2 \\ c_1 e - c_2 e^{-1} &= 2 \end{aligned}$, Solution is : $\left\{ c_2 = -\frac{2}{e+e^{-1}}, c_1 = \frac{2}{e+e^{-1}} \right\}$,

So $y(x) = \frac{2}{e+e^{-1}} e^x - \frac{2}{e+e^{-1}} e^{-x} - x$

2. a) let $u(r,\theta) = R(r)T(\theta)$

then $u_r = R'(r)T(\theta)$ $u_{rr} = R''(r)T(\theta)$ $u_{\theta\theta} = R(r)T''(\theta)$

and $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ becomes $R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) = 0$

$$r^2 R''(r)T(\theta) + rR'(r)T(\theta) = -R(r)T''(\theta)$$

$$\frac{r^2 R''(r) + rR'(r)}{-R(r)} = \frac{T''(\theta)}{T(\theta)} = k$$ since R and T are independent

resulting in the equations $r^2 R''(r) + rR'(r) + kR(r) = 0$

and $T''(\theta) - kT(\theta) = 0$

2. b) (i) $u_{xx}(x,t) = v_{xx}(x,t)$ $u_x(x,t) = v_x(x,t)$

$u_{tt}(x,t) = v_{tt}(x,t)$ $u_t(x,t) = v_t(x,t)$

$u(1,t) = 2$ and $u(x,t) - 2 = v(x,t) \Rightarrow v(1,t) = 0$

$u_x(0,t) = 0 \Rightarrow v_x(0,t) = 0$

$u(x,0) = -3\cos\frac{7\pi}{2} + 2$ and $u(x,t) - 2 = v(x,t) \Rightarrow v(x,0) = -3\cos\frac{7\pi}{2}$

2. b) (ii) let $v(x,t) = X(x)T(t)$

then $X''T = 9XT' \Rightarrow \frac{X''}{X} = 9\frac{T'}{T} = k$

resulting in the ordinary differential equations:

$X'' - kX = 0$ and $T' - \frac{k}{9}T = 0$

Boundary Conditions become: $X'(0)T(t) = 0$ and $X(1)T(t) = 0$

$$\Rightarrow X'(0) = 0 \text{ and } X(1) = 1$$

Solving the differential equation $X'' - kX = 0$ consider all values of k

$k < 0$ let $k = -u^2$; $u > 0$

$X'' + u^2X = 0$ has the solution: $X(x) = c_1 \cos ux + c_2 \sin ux$

and $X'(x) = -c_1 u \sin ux + c_2 u \cos ux$

B.C. $\Rightarrow X(1) = c_1 \cos u + c_2 \sin u = 0$ and $X'(0) = c_2 u = 0$

$\Rightarrow c_2 = 0$ thus $c_1 \cos u = 0 \Rightarrow u_n = \frac{(2n-1)\pi}{2}$ $n = 1, 2, \ldots$

$$\Rightarrow k_n = -\frac{(2n-1)^2\pi^2}{4} \quad n = 1, 2, \ldots$$

$$\text{so} \quad X_n(x) = c_n \cos \frac{(2n-1)\pi}{2}x$$

$k = 0 \Rightarrow X'' = 0$ which has the solution: $X(x) = c_1 x + c_2$ and $X'(x) = c_1$

B.C. $\Rightarrow X(1) = c_1 + c_2 = 0$ and $X'(0) = c_1 = 0 \Rightarrow c_2 = 0$

thus $X(x) \equiv 0$ is the trivial solution.

$k > 0$ let $k = u^2$; $u > 0$

$X'' - u^2X = 0$ has the solution: $X(x) = c_1 e^{ux} + c_2 e^{-ux}$

and $X'(x) = c_1 u e^{ux} - c_2 u e^{-ux}$

B.C. $\Rightarrow X'(0) = c_1 u - c_2 u = 0 \Rightarrow c_1 = c_2$

and $X(1) = c_1 e^u + c_2 e^{-u} = 0 \Rightarrow c_1 e^u + c_1 e^{-u} = 0 \Rightarrow c_1(e^u + e^{-u}) = 0$

$\Rightarrow c_1 = c_2 = 0$ thus $X(x) \equiv 0$ is the trivial solution.

Using the non-trivial solution $k_n = -\frac{(2n-1)^2\pi^2}{4}$ $X_n(x) = c_n \cos \frac{(2n-1)\pi}{2}x$,

the equation $T' - \frac{k}{9}T = 0$ becomes $T' + \frac{(2n-1)^2\pi^2}{36}T = 0$

solving by separating $\frac{T'}{T} = -\frac{(2n-1)^2\pi^2}{36} \Rightarrow \int \frac{T'}{T} = -\int \frac{(2n-1)^2\pi^2}{36}$

$\Rightarrow \ln T = -\frac{(2n-1)^2\pi^2}{36}t + c \Rightarrow T_n(t) = c_n e^{-\frac{(2n-1)^2\pi^2}{36}t}$

Therefore $v_n(x,t) = X_n(x)T_n(t)$

$$v_n(x,t) = c_n \cos \frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2\pi^2}{36}t}$$

$$\text{so} \quad v(x,t) = \sum_{n=1}^{\infty} c_n \cos \frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2\pi^2}{36}t}$$

Using I.C. to compute coefficients:

$$v(x,0) = \sum_{n=1}^{\infty} c_n \cos \frac{(2n-1)\pi x}{2} = -3 \cos \frac{7\pi x}{2}$$

by equating coefficients: $c_1 = 0, c_2 = 0, c_3 = -3, c_4 = 0, \dots$

$v(x,t) = -3\cos\frac{7\pi x}{2}\, e^{-\frac{49\pi^2}{36}t}$ is the solution.

2. b) (iii) $\quad u(x,t) = -3\cos\frac{7\pi x}{2}\, e^{-\frac{49\pi^2}{36}t} + 2$

3. a) $\quad A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

Use row operations on the matrix: $\begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{array}{r} R_2 \to R_1 \\ -3R_2 + R_3 \to R_3 \\ -2R_2 + R_1 \to R_2 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & -1 & -5 & 1 & 2 & 0 \\ 0 & -5 & -7 & 0 & -3 & 1 \end{bmatrix}$$

$$\begin{array}{r} -R_2 \to R_2 \\ -5R_2 + R_3 \to R_3 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 5 & -1 & -2 & 0 \\ 0 & 0 & 18 & -5 & 7 & 1 \end{bmatrix}$$

$$\begin{array}{r} \frac{1}{18}R_3 \to R_3 \\ \frac{-3}{18}R_3 + R_1 \to R_1 \\ \frac{-5}{18}R_3 + R_2 \to R_2 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & \frac{15}{18} & \frac{-3}{18} & \frac{-3}{18} \\ 0 & 1 & 0 & \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \\ 0 & 0 & 1 & \frac{-5}{18} & \frac{7}{18} & \frac{1}{18} \end{bmatrix}$$

$$-2R_2 + R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{18} & \frac{-5}{18} & \frac{7}{18} \\ 0 & 1 & 0 & \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \\ 0 & 0 & 1 & \frac{-5}{18} & \frac{7}{18} & \frac{1}{18} \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \\ -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{bmatrix}$$

3.b) Written in augmented matrix form the system
$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 + 3x_2 + 2x_3 + 4x_4 = 0$$
$$2x_1 + x_3 - x_4 = 0$$

becomes: $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 0 \\ 2 & 0 & 1 & -1 & 0 \end{bmatrix}$

reduce to row echelon form: $\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solution is $\{x_2 = x_2, x_4 = x_4, x_3 = -2x_2 - 3x_4, x_1 = x_2 + 2x_4\}$

or $\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$

4. a) $\int_C (3x - 3y)dx + x^2 dy \qquad C : \frac{x^2}{4} + \frac{y^2}{9} = 1$

Parameterize the ellipse: $x = 2\cos\theta \qquad dx = -2\sin\theta d\theta$

$y = 3\sin\theta \qquad dy = 3\cos\theta d\theta$

in the second quadrant $\frac{\pi}{2} \leq \theta \leq \pi$

$$\int_C (6\cos\theta - 9\sin\theta)(-2\sin\theta)d\theta + (2\cos\theta)^2 3\cos\theta d\theta$$
$$\int_{\frac{\pi}{2}}^{\pi} (-12\cos\theta\sin\theta + 18\sin^2\theta + 12\cos^3\theta)d\theta = \frac{9}{2}\pi - 2$$

b) Verify the divergence theorem for $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$

$\iint_{\partial G} \vec{v} \bullet \vec{n} dS$ using spherical coordinates ∂G becomes $\rho = a$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = a\sin\varphi\cos\theta\vec{i} + a\sin\varphi\sin\theta\vec{j} + a\cos\varphi\vec{k}$$

then $\vec{r}_\varphi = a\cos\varphi\cos\theta\vec{i} + a\cos\varphi\sin\theta\vec{j} - a\sin\varphi\vec{k}$

and $\vec{r}_\theta = -a\sin\varphi\sin\theta\vec{i} + a\sin\varphi\cos\theta\vec{j} + 0\vec{k}$

$$\vec{N} = \vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\varphi\cos\theta\vec{i} + a^2\sin^2\varphi\sin\theta\vec{j} + (a^2\cos\varphi\sin\varphi\cos^2\theta + a^2\cos\varphi\sin\varphi\sin^2\theta)\vec{k}$$

$$\vec{N} = a^2\sin^2\varphi\cos\theta\vec{i} + a^2\sin^2\varphi\sin\theta\vec{j} + a^2\cos\varphi\sin\varphi\vec{k}$$

at $(\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow \vec{N} = a^2\vec{j}$ which points outward.

$$\vec{v} = a\sin\varphi\cos\theta\vec{i} + a\sin\varphi\sin\theta\vec{j} + a\cos\varphi\vec{k}$$

$$\iint_{\partial G} \vec{v} \bullet \vec{n} dS = \int_0^{2\pi}\int_0^{\pi}(a^3\sin^3\varphi\cos^2\theta + a^3\sin^3\varphi\sin^2\theta + a^3\cos^2\varphi\sin\varphi)d\varphi d\theta = 4\pi a^3$$

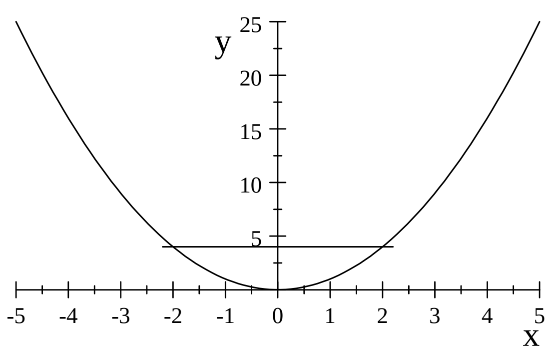
$$\iiint_G div\vec{v}dV \qquad div\vec{v} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

again using spherical coordinates

(or 3 times the volume of a sphere of radius a)

$$\iiint_G div\vec{v}dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a}(3\rho^2\sin\varphi)d\rho d\varphi d\theta = 4\pi a^3$$

5. a) (i) Graph the region of integration of $\int_{-2}^{2}\int_{x^2}^{4} x^2 y dydx$



$y = x^2$

$y = 4$

(ii) $\int_{-2}^{2}\int_{x^2}^{4} x^2 ydydx = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} x^2 ydxdy$

5. b) Verify Green's Theorem for $\oint_C (x^2+y^2)dx - 2xydy$

C is the triangle with vertices $(0,0),(1,0),(0,1)$

Let $C_1 : (0,0) \to (1,0)$

then $0 \le x \le 1$ and $y = 0 \Rightarrow dy = 0$

$$\oint_{C_1}(x^2+y^2)dx - 2xydy \Rightarrow \int_0^1 x^2 dx = \tfrac{1}{3}$$

$C_2 : (1,0) \to (0,1)$

then parameterize $y = t \Rightarrow dy = dt \qquad 0 \le t \le 1$

$$x = 1 - t \Rightarrow dx = -dt$$

$$\oint_{C_2}(x^2+y^2)dx - 2xydy = \int_0^1 -\left[(1-t)^2 + t^2\right]dt - 2(1-t)tdt$$

$$= \int_0^1 [-1 + 2t - 2t^2] - 2t + 2t^2 dt = \int_0^1 (-1)dt = -1$$

$C_3 : (0,1) \to (0,0)$

then $1 \ge y \ge 0$ and $x = 0 \Rightarrow dx = 0$

$$\oint_{C_3}(x^2+y^2)dx - 2xydy = \int_0^1 o dy = 0$$

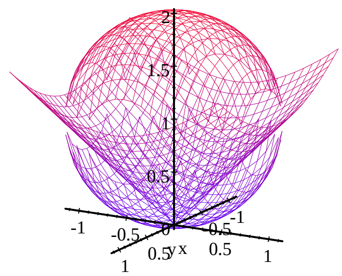
Finally:

$$\oint_C Pdx + Qdy = \oint_{C_1} Pdx + Qdy + \oint_{C_2} Pdx + Qdy + \oint_{C_3} Pdx + Qdy = -\tfrac{2}{3}$$

Let $P = (x^2+y^2)$ and $Q = -2xy$

then $\iint_G (Q_x - P_y)dA_{x,y} = \int_0^1 \int_0^{1-y} (-2y - 2y)dxdy = -\tfrac{2}{3}$

6. a) Find the volume of the region of the ball $x^2 + y^2 + (z-1)^2 = 1$ cut out by the cone $x^2 + y^2 = z^2$ using spherical coordinates.



$$x^2 + y^2 + (z-1)^2 = 1 \Rightarrow (\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 + (\rho\cos\varphi - 1)^2 = 1$$
$$\Rightarrow \rho^2 = 2\rho\cos\varphi \Rightarrow \rho = 2\cos\varphi$$
$$x^2 + y^2 = z^2 \Rightarrow (\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 = (\rho\cos\varphi)^2$$
$$\Rightarrow \tan^2\varphi = 1 \Rightarrow \varphi = \tfrac{\pi}{4}$$

Solution is : $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\cos\varphi} \rho^2 \sin\varphi d\rho d\varphi d\theta$

6. b) $\vec{F} = (2x - y - z)\vec{i} + (2y - x)\vec{j} + (2z - x)\vec{k}$

(i) $curl\vec{F} = \nabla \times (2x - y - z, 2y - x, 2z - x)$
$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$

(ii) Does there exist $\nabla\Phi = \vec{F}$? Yes, since $curl\vec{F} = \vec{0}$. Find Φ.

$$\Phi = \int(2x - y - z)dx = x^2 - xy - xz + g(y,z)$$
$$\Phi_y = -x + g_y = 2y - x \Rightarrow g_y = 2y \Rightarrow g(y,z) = \int 2y dy = y^2 + h(z)$$
$$\Phi = x^2 - xy - xz + y^2 + h(z)$$
$$\Phi_z = -x + h'(z) = 2z - x \Rightarrow h'(z) = 2z \Rightarrow h(z) = \int 2z dz = z^2 + c$$

Scalar potential is $\Phi = x^2 - xy - xz + y^2 + z^2 + c$

7. a) Evaluate $\iint_S \vec{F} \bullet \vec{n} dS$ where $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ and

S : $2x + 2y + z = 3$ in the first octant and $\vec{n} \bullet \vec{k} > 0$

Parameterize S let $z = 3 - 2x - 2y$

then $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + (3 - 2x - 2y)\vec{k}$

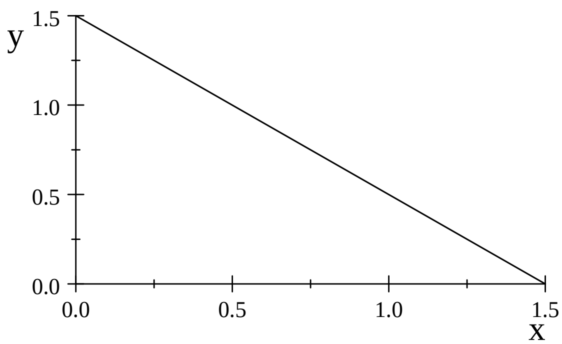
$$\vec{r}_x = 1\vec{i} + 0\vec{j} - 2\vec{k}$$
$$\vec{r}_y = 0\vec{i} + 1\vec{j} - 2\vec{k}$$

$$\vec{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix} = 2\vec{i} + 2\vec{j} + 1\vec{k}$$

$\vec{F} = x\vec{i} + y\vec{j} + (3 - 2x - 2y)\vec{k}$ parameterized

$\vec{F} \bullet \vec{N} = 2x + 2y - 3 - 2x - 2y = -3$

The projection of S into the xy-plane is $y = \frac{3}{2} - x$



Solution $\iint_S \vec{F} \bullet \vec{n}\,dS = \int_0^{\frac{3}{2}} \int_0^{\frac{3}{2}-y} -3\,dxdy$

Remember: $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ the unit normal and $dS = |\vec{r}_u \times \vec{r}_v|\,dA_{u,v}$

so $\vec{n}dS = (\vec{r}_u \times \vec{r}_v)\,dA_{u,v}$ I'm calling $\vec{r}_u \times \vec{r}_v = \vec{N}$

7. b) The volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$

let $z = \pm\sqrt{a^2 - x^2}$ and $y = \pm\sqrt{a^2 - x^2}$ then volume is the triple

integral $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dzdydx = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{(a^2 - x^2)}\,dydx$

$= \int_{-a}^{a} 4(a^2 - x^2)dx = \frac{16}{3}a^3$

8. a) State Stoke's Theorem

$$\iint_{\vec{S}} (\nabla \times \vec{v}) \bullet \vec{n}dS = \int_{\partial\vec{S}} \vec{v} \bullet d\vec{r}$$

8. b) Evaluate $\oint_C (2ydx - 2xdy + z^2xdz)$ where $C : x^2 + y^2 = 1, \quad z = 5$

This is the right hand side of Stoke's Theorem, and we know that

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ so $d\vec{r} = dx\vec{i} + dy\vec{j} + d\overrightarrow{zk},$ and we see from

the given expression that $\vec{v} = 2y\vec{i} - 2x\vec{j} + z^2x\vec{k}$

$$curl\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -2x & z^2x \end{vmatrix} = 0\vec{i} - z^2\vec{j} + (-2-2)\vec{k} = 0\vec{i} - z^2\vec{j} + -4\vec{k}$$

Given $z = 5$ we write $\vec{r} = x\vec{i} + y\vec{j} + 5\vec{k}$

so $\vec{r}_x = 1\vec{i} + 0\vec{j} + 0\vec{k}$ and $\vec{r}_y = 0\vec{i} + 1\vec{j} + 0\vec{k}$

$$\vec{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$$

Then $curl\vec{v} \bullet \vec{n}dS = curl\vec{v} \bullet \vec{N}dA_{x,y} = -4dA_{x,y}$

Note: $\vec{S}$ is the interior of the circle, so integrating over the circle

$$\iint_{\vec{S}} \left(\nabla \times \vec{v}\right) \bullet \vec{n}dS = \iint_{\vec{S}} -4dA_{x,y} = -4 \text{ (Area of circle } r = 1\text{)} = -4\pi$$

or $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} -4dxdy = -4\pi$

8. c) Show that $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = (\alpha-\beta)(\gamma-\alpha)(\beta-\gamma)$

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = \beta\gamma^2 - \beta^2\gamma - \alpha\gamma^2 + \alpha^2\gamma + \alpha\beta^2 - \alpha^2\beta$$

$$= -\alpha\gamma^2 - \alpha^2\beta + \alpha^2\gamma - \beta^2\gamma + \beta\gamma^2 + \beta^2\alpha + (\alpha\beta\gamma - \alpha\beta\gamma)$$

$$= \alpha\beta\gamma - \alpha^2\beta - \beta^2\gamma + \beta^2\alpha - \alpha\gamma^2 + \alpha^2\gamma + \beta\gamma^2 - \alpha\beta\gamma$$

$$= (\alpha\gamma - \alpha^2 - \beta\gamma + \beta\alpha)(\beta-\gamma) = (\alpha-\beta)(\gamma-\alpha)(\beta-\gamma)$$

9. a) Solve the eigenvalue problem:

$$y'' + 2y' + (\lambda+1)y = 0, \quad y(0) = y(1) = 0$$

The characteristic equation is $r^2 + 2r + (\lambda + 1) = 0$

so $r = \dfrac{-2 \pm \sqrt{4 - 4(\lambda + 1)}}{2} = -1 \pm \sqrt{-\lambda}$

There are three cases depending on the value of λ.

Case 1: $\lambda < 0$ let $\lambda = -u^2 \quad u > 0$

then $r = -1 \pm \sqrt{-(-u^2)} = -1 \pm u$

$y(x) = c_1 e^{(-1+u)x} + c_2 e^{(-1-u)x}$

B.C. $\Rightarrow y(0) = c_1 e^{(-1+u)0} + c_2 e^{(-1-u)0} = 0 \Rightarrow c_1 + c_2 = 0$

and $y(1) = c_1 e^{(-1+u)1} + c_2 e^{(-1-u)1} = c_1 e^{-1+u} + c_2 e^{-1-u} = 0$

$\Rightarrow c_1 e^{-1+u} - c_1 e^{-1-u} = 0 \Rightarrow c_1 e^u - c_1 e^{-u} = 0 \Rightarrow c_1(e^u - e^{-u}) = 0$

$\Rightarrow c_1 = 0 = c_2$

thus $y(x) \equiv 0$ the trivial solution $\Rightarrow$ no eigenvalues

Case 2: $\lambda = 0$

then $r = -1$

$y(x) = c_1 e^{-x} + c_2 x e^{-x}$

B.C. $\Rightarrow y(0) = c_1 e^0 + c_2(0)e^0 = 0 \Rightarrow c_1 = 0$

and $y(1) = c_1 e^{-1} + c_2(1)e^{-1} = 0 \Rightarrow c_1 + c_2 = 0$

$\Rightarrow c_1 = 0 = c_2$

thus $y(x) \equiv 0$ the trivial solution $\Rightarrow$ no eigenvalues

Case 3: $\lambda > 0$ let $\lambda = u^2 \quad u > 0$

then $r = -1 \pm \sqrt{-(u^2)} = -1 \pm ui$

$y(x) = e^{-x}(c_1 \cos ux + c_2 \sin ux)$

B.C. $\Rightarrow y(0) = e^0(c_1 \cos 0 + c_2 \sin 0) = 0 \Rightarrow c_1 = 0$

and $y(1) = e^{-1}(c_1 \cos u + c_2 \sin u) = 0$

$\Rightarrow c_2 \sin u = 0 \Rightarrow u = n\pi \quad n = 1, 2, \ldots$

$\Rightarrow$ eigenvalues $\lambda_n = n^2\pi^2 \quad n = 1, 2, \ldots$

eigenfunctions $y_n(x) = c_n e^{-x} \sin n\pi x$

9. b) Find the eigenvalues of the matrix $\vec{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

We wish to solve the matrix equation $\left(\vec{A} - r\vec{I}\right)\vec{u} = \vec{0}$

This has non-trivial solutions if and only if $\det\left(\vec{A} - r\vec{I}\right) = 0$

$$\begin{vmatrix} 1-r & -1 & 0 \\ 0 & 2-r & 1 \\ 0 & 0 & -1-r \end{vmatrix} = (1-r)(2-r)(-1-r) = 0$$

$\Rightarrow r = -1, 1, 2$ are the eigenvalues

Eigenvectors corresponding to the eigenvalues are:

$r = -1$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In augmented matrix form: $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$u_2 = -\frac{1}{3}u_3$ and $u_1 = \frac{1}{2}u_2 = -\frac{1}{6}u_3$

$$\vec{u} = s\begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$$

$r = 1$

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In augmented matrix form: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$

$u_1 = s$ is arbitrary $\quad u_3 = 0 \quad u_2 = -u_3 = 0$

$$\vec{u} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$r = 2$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In augmented matrix form: $\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

$u_3 = 0 \qquad u_1 = -u_2$

$$\vec{u} = s\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$